A NOTE ON THE PRECEDING PAPER

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The purpose of this note is to identify the abstract F. and M. Riesz theorem of the preceding paper [2] (in the "compact-continuous" case) with that of [1], via a simple measure theoretic consequence of an extension of von Neumann's Minimax Theorem.

The cited theorems of [1], [2] make assertions about the components of a measure μ relative to two a priori distinct Lebesgue (-like) decompositions. One is given a w^*-compact convex set $M = M_\varphi(A) = M(A, \varphi)$ of (Baire) probability measures on a compact Hausdorff space X; in [1] the decomposition has the form

$$\mu = \mu_{F'} + \mu_F \tag{1}$$

where $\mu_{F'}$ vanishes on the common null sets of the elements of M while F is such a common null set: $\lambda(F) = 0$, all $\lambda \varepsilon M$. In [2], with μ_λ the part of μ absolutely continuous with respect to λ, μ_λ' the singular component, the decomposition has the form

$$\mu = \mu_\lambda + \mu_\lambda' \tag{2}$$

where $\lambda \varepsilon M$ is chosen so that $||\mu_\lambda||$ is a maximum, and thus so that μ_λ' and each $\lambda' \varepsilon M$ are mutually singular. (If $||\mu_{\lambda_n}|| \to c = \sup_{\lambda \varepsilon M} ||\mu_\lambda||$, then $c = ||\mu_{\lambda_0}||$ for $\lambda_0 = \sum_{n=1}^{\infty} 2^{-n}\lambda_n$. If $(\mu_{\lambda_0}')_\lambda$ were to be non-zero for some $\lambda \varepsilon M$, we would have $||\mu_{\frac{1}{2}(\lambda+\lambda_0)}|| > c$, so that μ_{λ_0}' is λ-singular for all $\lambda \varepsilon M$.) Since μ_F in (1) is clearly singular with respect to each λ in M, coincidence of (1) and (2) will follow from the result below which asserts that if μ is λ-singular for all $\lambda \varepsilon M$, then there is a common null set F of the elements of M such that $\mu = \mu_F$. This strengthens the cited results of [1] and [2], in the first instance giving a more usable form to the "M-absolutely continuous" component of μ, and in the second case by showing the singular component of μ is carried by a single common null set.

Theorem. *Let X be a compact Hausdorff space, and let M be a w^*-compact convex set of Baire probability measures on X. Let μ be a Baire measure on X such that μ is λ-singular for all $\lambda \varepsilon M$. Then there is a Baire set F such that $\lambda(F) = 0$ for all $\lambda \varepsilon M$ and $\mu = \mu_F$.*

Proof. Since $|\mu|$ (the variation of μ) is λ-singular for all $\lambda \varepsilon M$, we may assume $\mu \geq 0$. If $\lambda \varepsilon M$ and μ are regarded as functionals in $C^R(X)^*$, then their mutual singularity states that as functionals $\mu \wedge \lambda = 0$. For $0 \leq f \varepsilon C^R(X)$, $(\mu \wedge \lambda)(f) = \inf \{\mu(f - g) + \lambda(g) : g \varepsilon C^R(X), 0 \leq g \leq f\}$. In particular,

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