IDENTITIES FROM GRAPHS

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1. Introduction. Let G be a finite graph. Let P be the set of partitions of $V(G)$ (the vertex set of G) into two parts. If π belongs to P, we write $\pi = (p, \bar{p})$ where p and $\bar{p}$ are the parts of π. We further use the notation G_S $(S \subseteq V(G))$ to denote the subgraph of G whose vertex set is S and whose edge set consists of those edges of G both of whose ends are in S (we call this subgraph the restriction of G to S). Thus to every $\pi \varepsilon P$ there corresponds a pair of subgraphs $(Gp, G\bar{p})$ of G. For every $\pi \varepsilon P$ let us use the symbol $B(G, \pi) = B(\pi)$ to denote the number of edges E in G such that one end of E belongs to p and the other to $\bar{p}$; we also use $F_2(\pi) = F_2(G, \pi)$ to denote $C(Gp)C(G\bar{p})$, where $C(H)$, the complexity of H, is the number of spanning trees in the graph H. Then it is proved in [1] that

$$\sum_{\pi \varepsilon P} B(\pi)F_2(\pi) = (\alpha_0(G) - 1)C(G), \tag{1}$$

where $\alpha_0(G) = |V(G)|$. Identity (1) was then used in connection with certain graphs to obtain combinatorial identities. It is our purpose here to obtain the most general identity possible by use of graphs in connection with (1). The identity will be given in a determinantal form which requires some preliminary discussion.

2. Admissible matrices and associated functions. Let $A = (a_{ij})$, where $i, j = 1, 2, \cdots, n$, be a complex *symmetric* matrix such that $\sum_{j=1}^{n} a_{ij} = 0$ $(i = 1, 2, \cdots, n)$. Such a matrix will be called admissible. By symmetry, it follows that $\sum_{i=1}^{n} a_{ij} = O(j = 1, 2, \cdots, n)$, and under such circumstances it may easily be demonstrated that all n^2 cofactors of A are equal. We shall use $C(A)$ to denote the common cofactor value of such a matrix. By definition, $C(0) = 1$. Further, if p is any subset of $S_n = \{1, 2, \cdots, n\}$, we may form a new admissible matrix $F(A, p) = (f_{ij})$, with $i, j \varepsilon p$, by defining

$$\begin{aligned} f_{ij} &= a_{ij} \ (i \neq j) \\ f_{ii} &= a_{ii} + \sum_{j \varepsilon \bar{p}} a_{ij} \end{aligned} \tag{2}$$

where $\bar{p}$ is the complement of p in S_n. Now, for any admissible $n \times n$ matrix A and any element $\pi = (p, \bar{p})$ of the set of bipartitions of S_n, we define $B(A, \pi) = B(\pi)$ and $F_2(A, \pi) = F_2(\pi)$ by

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