ANALYTIC FUNCTIONS OF AN INFINITE NUMBER OF COMPLEX VARIABLES

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Introduction. Most of the applications of the theory of analytic functions of several complex variables to the study of Banach algebras are based ultimately on the following situation. Let $\mathfrak{A}$ be a complex, commutative Banach algebra (with 1) and let $\Phi_{\mathfrak{A}}$ be its carrier space (space of maximal ideals) regarded as consisting of homomorphisms, $\varphi : a \rightarrow \hat{a}(\varphi)$, of $\mathfrak{A}$ onto the complex numbers C. Endowed with the weakest topology under which each of the functions $\hat{a}$ is continuous, $\Phi_{\mathfrak{A}}$ is a compact Hausdorff space. If $\{z_\lambda : \lambda \in \Lambda\}$ is an arbitrary subset of $\mathfrak{A}$, then $\varphi \rightarrow \{\hat{z}_\lambda(\varphi)\}$ defines a continuous mapping of $\Phi_{\mathfrak{A}}$ into the cartesian product C^Λ of "Λ" complex planes. The image, $\mathrm{Sp}(z_\lambda : \lambda \in \Lambda)$, of $\Phi_{\mathfrak{A}}$ in C^Λ is compact and is called the *joint spectrum* of the elements z_λ. If $\{z_\lambda : \lambda \in \Lambda\}$ is a system of generators for $\mathfrak{A}$, then the mapping is a homeomorphism, $\mathrm{Sp}(z_\lambda : \lambda \in \Lambda)$ is a polynomially convex set in C_Λ, and each function $\hat{a}$ is a uniform limit on $\mathrm{Sp}(z_\lambda : \lambda \in \Lambda)$ of polynomials [8; 151]. If the system of generators happens to be finite, then C^Λ is a finite dimensional complex space so results from several complex variables are immediately available for the study of $\mathfrak{A}$. When $\mathfrak{A}$ is not finitely generated, then the usual technique is to reduce to the finite dimensional case by application of a "lemma" due to Arens and Calderón [3, Theorem 2.3]. This general state of affairs suggests both the possibility and desirability of a systematic extension of some of the fundamental results for n complex variables to functions of "Λ" complex variables. This is the purpose of the present paper. Such results are not only technically useful in applications to Banach algebras, but are obviously interesting in their own right.

In §1 a few definitions and elementary properties concerning polynomially convex sets in C^Λ are obtained. In §2 we define a function to be holomorphic in C^Λ if it is locally a uniform limit of polynomials. This condition, which gives ordinary holomorphic functions when Λ is finite, is quite restrictive in the infinite case. In fact, a holomorphic function in C^Λ (defined on an open set) depends locally on only a finite number of the variables. Thus, locally, it is essentially an ordinary holomorphic function. Nevertheless, this class of functions appears to be precisely what is needed for the type of result that we wish to obtain here. For example, the Oka theorem on polynomial approximation of holomorphic functions and extension theorems for holomorphic functions, also due essentially to Oka, generalize more-or-less immediately to infinite dimensions (Theorems 2.3, 2.4, 2.5). A theorem such as the polynomial approximation theorem (Theorem 2.3) could also be regarded as a "union", so-to-speak, of the corresponding theorems for C^π, where π ranges over all

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