CONNECTIVITY, SEMI-CONTINUITY, AND THE DARBOUX PROPERTY

By Jack B. Brown

Introduction. This paper is concerned with certain topological properties of the graphs of real functions. Only real functions will be considered, and the word graph will always refer to the graph of a real function. If f is a point set in the plane, then the X-projection of f is the set of all abscissas of points of f. Certain relationships will be established between the property of connectivity and the following definition: if f and g are graphs, then the statement that g *cuts* f means that g has X-projection an interval and there are two points P and Q of f, P higher than Q, such that (1) the abscissas of P and Q are in the X-projection of g, (2) every point of Cl (g) is lower than P and higher than Q, and (3) f and Cl (g) do not intersect. If f is a graph with X-projection an interval, then it is clear that f has the Darboux property if and only if no subset of a horizontal line cuts f. It is well known [1] that there is a disconnected graph which has the Darboux property. An example will be given of a disconnected graph with X-projection the unit interval I which no continuous graph cuts. On the other hand, the following theorem will be proved:

Theorem. *If f is a graph with X-projection I and no lower semi-continuous graph cuts f, then f is connected.* An example of an application of this theorem will be given.

Example 1. Let C be the collection of all continuous graphs with X-projection I, and let H be a collection of mutually exclusive, countable, dense subsets of I such that the union of all the sets in H is I. There is a reversible transformation T from H onto C. Let M be a strictly increasing graph with X-projection I which is continuous from the right and discontinuous only at points with rational abscissa in (0, 1) and $M(0) = 0$ and $M(1) = 1$. Let N be the set of all points Z such that Z is a point of M or a point of a vertical interval with endpoints on Cl (M). The graph f is defined as follows: if x is in I and h is the set in H which contains x and $g = T(h)$, then $f(x) = g(x)$ if the point $(x, g(x))$ is not on N and $f(x) = -1$ otherwise. Since f and N do not intersect, but f contains points above N and points below N, then f is not connected. It will now be shown that it is not only true that no continuous graph cuts f, but if g is a continuous graph with X-projection a subinterval of I, then f and g have c-many points in common, where c denotes the cardinality of the continuum. Suppose g is a continuous graph with X-projection a subinterval I' of I. g cannot be a subset of N, so it follows that there is a proper subinterval I'' of I' such that g'', the subset of g which has X-projection I'', and N do not intersect. Now, g'' is a subset of c-many different graphs in C and since the sets in H are mutually

Received November 22, 1967.