L_p AVERAGES AND THE NONLINEAR RENEWAL EQUATION

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1. Introduction and statement of results. Let ψ be a continuous monotone function on $(0, \infty)$, and let G be a (right continuous) probability distribution on $[0, \infty)$. Suppose f and x are functions on $(-\infty, \infty)$ which are measurable, positive, bounded on finite intervals, and identically 1 on $(-\infty, 0)$. Define the operator $\mathfrak{M}$ by

$$(1) \qquad (\mathfrak{M}f)(t) = \psi^{-1}\left\{\int_0^\infty \psi[f(t-y)]\, dG(y)\right\},$$

and consider the equation

$$(2) \qquad x(t) = a + (\mathfrak{M}x)(t) \qquad (t \geq 0)$$

for $a > 0$.

When $\psi(x) = x$, $\mathfrak{M}$ is linear and (2) is the classical renewal equation. In this paper we study nonlinear operators, and prove the analogue of the weak renewal theorem for them (see Theorem 2). We also investigate existence and uniqueness of solutions (Theorem 1), borderline cases (Theorem 3) and various pathologies (Theorem 4). Part of Theorem 1 is due to Chover and Ney [1], as is a special case of Theorem 2. We assume throughout that $G(0) < 1$.

THEOREM 1. $\exists_1$ *solution to* (2) *provided either*

(3) ψ^{-1} *is convex, or*

(4) ψ *is increasing,* $\psi(0+) = 0$, *and* $\mathfrak{M}$ *satisfies Minkowski's inequality*:

$$f \geq 0, \qquad g \geq 0 \text{ implies } \mathfrak{M}(f+g) \leq \mathfrak{M}f + \mathfrak{M}g.$$

This solution is monotone nondecreasing and $\to \infty$ *as its argument* $\to +\infty$.

Let p be a real number. For $p \neq 0$, define $\mathfrak{M}_p$ by

$$(\mathfrak{M}_p f)(t) = \left\{\int_0^\infty [f(t-y)]^p\, dG(y)\right\}^{1/p}.$$

Define $\mathfrak{M}_0$ by

$$(\mathfrak{M}_0 f)(t) = \exp\left\{\int_0^\infty \log\,[f(t-y)]\, dG(y)\right\}.$$

$\mathfrak{M}_1$ is the arithmetic mean, $\mathfrak{M}_0$ the geometric mean, and $\mathfrak{M}_{-1}$ the harmonic mean.

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