QUASICONFORMALLY EQUIVALENT CURVES

By Seppo Rickman

1. Introduction. We call two Jordan curves γ and γ' lying in the extended complex plane $\mathcal{C}^* = \mathcal{C} \cup \{\infty\}$ *quasiconformally equivalent* if there exists a quasiconformal mapping of $\mathcal{C}^*$ onto itself which maps γ onto γ'. This terminology is due to Springer [7] who introduced a slightly more general definition concerning the equivalence of curve systems. Those Jordan curves which are quasiconformally equivalent with circles, called quasiconformal curves or quasicircles, form an important equivalence class extensively studied recently, see e.g. [2], [3], [5]–[7]. This paper is devoted to the study of quasiconformal equivalence of Jordan curves in the general case.

Two Jordan curves γ and γ' are obviously quasiconformally equivalent if and only if there exists a homeomorphism $\varphi : \gamma \rightarrow \gamma'$ which can be extended to a quasiconformal mapping of $\mathcal{C}^*$ onto itself. If this is the case, we say briefly that φ has a quasiconformal extension. It follows from an extension theorem of Lehto and Virtanen [5, Satz II.8.1] that if φ can be extended to a quasiconformal mapping of a neighborhood of γ, then it can be extended quasiconformally to the whole plane. As our main result we establish by means of cross-cuts of the complementary domains of the curves and by cross-ratios a necessary and sufficient condition for such a homeomorphism φ to have a quasiconformal extension (Theorem 2). The proof of sufficiency is divided into two steps: In §3 an extension of φ is obtained which is quasiconformal in the complementary domains of γ. In §4 it is proved that such an extension under the given condition on φ actually is quasiconformal in the whole plane.

In the last part we prove that a homeomorphism of a Jordan curve onto another which has local quasiconformal extensions has a quasiconformal extension. In addition to the result of §3 modified to closed Jordan arcs, a removability theorem (Theorem 3) of §4 is used in the proof.

2. Necessary conditions and the main theorem. Let γ and γ' be two Jordan curves in $\mathcal{C}^*$ and $\varphi : \gamma \rightarrow \gamma'$ a homeomorphism. Let (z_1, z_2, z_3, z_4) denote the cross-ratio of the sequence z_1, z_2, z_3, z_4 of points in $\mathcal{C}^*$. For finite distinct points we have then

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

The absolute value of (z_1, z_2, z_3, z_4) is denoted by $|z_1, z_2, z_3, z_4|$.

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