INCIDENCE FUNCTIONS AS GENERALIZED ARITHMETIC FUNCTIONS, III.

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1. Introduction. As the title indicates, this paper is the third in a series [10], [11] devoted to the study of incidence algebras of locally finite partially ordered sets by generalizing properties of the familiar algebra of arithmetic functions. The emphasis in this paper will be different from that of its predecessors, however. In [10], [11] we concentrated on developing properties of incidence algebras in rather general contexts. As a byproduct, we obtained some generalizations and uniform methods of proof for algebras of arithmetic functions with various product operations other than the usual one, but the partially ordered sets under consideration in those applications were very special cases. In this paper we will limit ourselves essentially to these very special cases, that is, to a context which is much too specialized to say anything about incidence algebras in general, and yet is still very general from the point of view of arithmetic functions.

The main theorems in the paper contain generalizations of the Brauer-Rademacher identity and other closely related identities involving the Möbius and Euler functions. Many papers have been written giving generalizations and/or alternate proofs of the Brauer–Rademacher identity (see for example [4], [5], [6], [12], [13] and other references contained in some of these). The principal ideas used here are motivated by [12] and [6], and to a lesser extent, [4].

In [6], Cohen gave extensive generalizations of identities of Brauer-Rademacher [2], Hölder [8], and Landau [9], and demonstrated the interrelationships among them. (The generalized Hölder identity is due to Anderson and Apostol [1], but Cohen's proof is different from theirs.) More recently, Subbarao [12] gave a very short proof of a generalization of the Brauer–Rademacher identity which appears to be quite different in form from Cohen's. It is perhaps worth mentioning here that Cohen's formula [6, (4.6)] is a special case of Subbarao's [12, (5)], since Subbarao does not mention this among his applications, nor does Cohen mention it in his review of Subbarao's paper (*Math. Rev.* 30 (1965) 3863).

Since the context of [12] is more general than that of [6], we use the approach of the former in presenting the main theorems (§4), generalizing not only the Brauer–Rademacher, but also the Hölder and Landau identities, as well as a summation formula of Cohen's [4, Theorem 7] which he derived from an equivalent of the Brauer–Rademacher identity. In §5 we specialize to the type of formulas appearing in [6], and in the concluding section we give some rather different special cases suggested by [12, §4].

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