AN INTERSECTION THEOREM FOR SETS OF CONSTANT WIDTH

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1. Introduction. Let K be a compact, convex subset of n-dimensional Euclidean space E_n. Let $h(K)$ be the smallest cardinal r having the following property: if $\mathfrak{F}$ is any pairwise intersecting family of translates of K, then there exist r points such that each member of $\mathfrak{F}$ contains at least one of them. Grünbaum [4] proved that for plane centrally symmetric convex sets K, $h(K) \leq 3$, and conjectured that this holds for any plane convex body. It is proved in [1] that $h(K) \leq 4$ for plane convex sets, but Grünbaum's conjecture remains unsettled. In this paper, we prove the following,

Theorem 1. *Let K be a compact convex plane set having constant width. Then $h(K) \leq 3$.*

It is known [2] that $h(K)$ can be equivalently defined as the least cardinal r having the following property: if Q is any set such that each pair of points of Q is contained in some translate of K, then Q is contained in the union of some r translates of K.

If K has constant width 1, then the condition that each pair of points of Q is contained in a translate of K is equivalent to Q having diameter at most 1. Hence Theorem 1 is a consequence of the following,

Theorem 2. *Let $Q \subset E_2$ have diameter 1, and let K be a plane convex set of constant width λ, where $\lambda \geq .9101$. Then Q is contained in the union of some 3 translates of K.*

It will be seen that the crux of the matter is the fact that any set of constant width .9101 contains a translate of a certain fixed pentagon (see the remarks at the beginning of the proof of Theorem 2). Theorems 1 and 2 are consequences of this fact. Indeed, it even follows that any plane set of diameter 1 can be covered by translates of any 3 given sets of constant width .9101.

The bound .9101 in Theorem 2 is certainly not the best possible. On the other hand it can be computed directly that a circular disk of diameter 1 cannot be covered by 3 Reuleaux triangles of width less than .89.

Theorem 2 incidentally settles a conjecture of Grunbaum [5; 277], and is proved in §3. In §2 we develop some of the machinery needed in the proof.

2. Some useful lemmas. By a *diameter* of a plane convex set we mean any closed line segment joining a pair of boundary points lying in parallel supporting lines. The endpoints of any diameter shall be referred to as *opposite* points.

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