SOME REMARKS ON THE ENUMERATION OF SYMMETRIC MATRICES

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Professor Carlitz [1] has studied the number, $S_n(r)$, of $n \times n$ symmetric integral matrices (a_{ij}) satisfying

$$\sum_{j=1}^{n} a_{ij} = r \qquad (1 < i < n). \tag{1}$$

He evaluated $S_n(1)$ and $S_n(r)$ for $1 \leq n \leq 4$.

H. Gupta [2] has considered the numbers $S_n(2)$ and obtained the recurrence

$$S_{n+1} = (2n + 1)S_n - (n)_2\{S_{n-1} + S_{n-2}\} + \frac{(n)_3}{2} S_{n-3} , \tag{2}$$

where for brevity we write $S_n = S_n(2)$ and $(n)_j = n(n - 1) \cdots (n - j + 1)$.

Here we continue this problem but note that it is convenient to replace (1) by the more general

$$\sum_{j=1}^{n} a_{ij} = r_i \qquad (1 \leq i \leq n), \tag{3}$$

where the r_i are arbitrary non-negative integers.

Let $S(r_1, \cdots , r_n)$ denote the number of $n \times n$ symmetric integral matrices (a_{ij}) satisfying (3). It is immediate that $S(r_1, \cdots , r_n)$ is symmetric in the variables r_i, and it is not difficult to show that $S(r_1, \cdots , r_n)$ is the coefficient of $x_1^{r_1} \cdots x_n^{r_n}$ in the expansion of

$$\prod_{j=1}^{n} (1 - x_i)^{-1} \prod_{i<j} (1 - x_i x_j)^{-1} \tag{4}$$

which we denote by $G(x_1, \cdots , x_n)$.

Writing (4) in the form

$$(1 - x_1)G(x_1 , \cdots , x_n) = G(x_2 , \cdots , x_n) \prod_{j=2}^{n} (1 - x_1 x_j)^{-1},$$

we obtain the recurrence

$$S(k_1 + 1, k_2, \cdots , k_n) = S(k_1, k_2, \cdots k_n) + \Sigma\, S(k_2 - j_2, \cdots , k_n - j_n), \tag{5}$$

where the sum extends over all $(n - 1)$-tuples $(j_2, \cdots , j_n)$ of non-negative integers satisfying $j_2 + \cdots + j_n = k_1 + 1$.

Received March 23, 1968. Supported in part by National Science Foundation grant GP-6382.