LEBESGUE DECOMPOSITION AND WEAKLY BOREL MEASURES

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Introduction. We prove in §1 a slight extension of the form of the Lebesgue Decomposition Theorem given by [6, Theorem 2.1]. We apply it to the study of regular and anti-regular weakly Borel and weakly Baire measures in §2 where these terms are defined. In §3 and §4 we apply it to the decomposition of weakly Borel and weakly Baire measures into regular and anti-regular parts. For example, we show (Theorem 3.6) that every σ-finite weakly Borel measure can be uniquely represented as the sum of a regular and an anti-regular weakly Borel measure. Finally, in §5 we introduce a slightly weaker form of anti-regularity and use it to investigate a similar type of decomposition.

1. Lebesgue decomposition. Let X be a set, $\mathcal{S}$ be a σ-ring of subsets of X, and μ, ν be measures on $\mathcal{S}$. ν is *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$, if $\nu(E) = 0$ whenever $E \in \mathcal{S}$ and $\mu(E) = 0$ [4; 124]. Following [6], we say that ν is *S-singular* with respect to μ, denoted $\nu S \mu$, if for each $E \in \mathcal{S}$ there is a set $F \in \mathcal{S}$ such that $F \subset E$, $\nu(F) = \nu(E)$, and $\mu(F) = 0$. A set $A \subset X$ is said to be *locally measurable* if $A \cap E \in \mathcal{S}$ for every $E \in \mathcal{S}$ [1; 35]. As is customary [4; 126], we say that ν is *singular* with respect to μ, denoted $\nu \perp \mu$, if there is a locally measurable set A such that $\nu(E \cap A) = 0 = \mu(E - A)$ for every $E \in \mathcal{S}$.

It is easy to see that although singularity is a symmetric relation, S-singularity does not possess this property. Moreover, clearly $\nu \perp \mu$ implies that $\nu S \mu$ and $\mu S \nu$. However, the converse does not hold. This is shown in [6] which also contains the following easily verified properties of S-singularity that we shall need.

I. $\nu S \nu$ if, and only if, $\nu = 0$.
II. If $\nu S \mu$ and $\lambda \ll \mu$, then $\nu S \lambda$.
III. If $\nu S \mu_1$ and $\nu S \mu_2$, then $\nu S(\mu_1 + \mu_2)$.
IV. If $\nu_1 S \mu$ and $\nu_2 S \mu$, then $(\nu_1 + \nu_2) S \mu$.
V. If $\nu \leq \mu + \lambda$ and $\nu S \lambda$, then $\nu \leq \mu$.

The following slight extension of [6, Theorem 2.1] will be basic throughout this paper.

Theorem 1.1 *Let μ and ν be measures on a σ-ring $\mathcal{S}$. Then there exists a unique decomposition $\nu = \nu_1 + \nu_2$ of ν into the sum of measures ν_1, ν_2 on $\mathcal{S}$ such that $\nu_1 \ll \mu$, $\nu_2 S \mu$, and $\nu_1 S \nu_2$.*

Remark. Note that $\nu_2 S \nu_1$ by Property II.

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