LAGUERRE SPACES OF ENTIRE FUNCTIONS

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We are concerned with examples of Hilbert spaces whose elements are entire functions and which have these properties.

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z - \bar{w})/(z - w)$ is in the space and has the same norm as $F(z)$.

(H2) For every nonreal number w, the linear functional defined on the space by $F(z) \to F(w)$ is continuous.

(H3) The function $F^*(z) = \bar{F}(\bar{z})$ belongs to the space whenever $F(z)$ belongs to the space and it has the same norm as $F(z)$.

These spaces are associated with entire functions $E(z)$ which satisfy the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for $y > 0$. If $E(z)$ is such a function, we write

$$E(z) = A(z) - iB(z)$$

where $A(z)$ and $B(z)$ are entire functions which are real for real z, and

$$K(w, z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z - \bar{w})].$$

Let $\mathcal{H}(E)$ denote the set of entire functions $F(z)$ such that

$$||F(t)||^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 \, dt < \infty,$$

and such that

$$|F(w)|^2 \leq K(w, w)\, ||F(t)||^2$$

for every complex number w. Then $\mathcal{H}(E)$ is a Hilbert space of entire functions which satisfies (H1), (H2), and (H3). For each complex number w, $K(w, z)$ belongs to $\mathcal{H}(E)$ as a function of z and

$$F(w) = \langle F(t), K(w, t)\rangle$$

for every $F(z)$ in $\mathcal{H}(E)$. A Hilbert space whose elements are entire functions, which satisfies (H1), (H2), and (H3), and which contains a nonzero element is equal isometrically to a space $\mathcal{H}(E)$.

Spaces of this kind related to Laguerre polynomials have been studied by de Branges [1]. Such spaces have these characteristic properties, which are stated for a fixed index ν: The function $zF(z + 1)$ belongs to the space whenever $F(z)$

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