NORMAL FORMS OF LOCAL DIFFEOMORPHISMS ON THE REAL LINE

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Let R be the real line with the coordinate x. By a local C^r diffeomorphism on R, we mean a C^r homeomorphism T from a neighborhood of the origin 0 onto another leaving 0 fixed.

Main Theorem. *Let T be a C^∞ local diffeomorphism on R. If T is orientation preserving (or reversing), and if the power series expansion of $T(x)$ is not equal to x (or if T is not formally equivalent to a reflection about 0), then given any $r < \infty$, there exists a C^r local diffeomorphism σ_1 on R such that $T_1 = \sigma_1^{-1} T \sigma_1$ is in one of the following normal forms*:

(i) $T_1(x) = \lambda x, |\lambda| \neq 1, \lambda > 0$ (*or* < 0);
(ii) $T_1(x) = x + (\pm x)^{\nu+1} + cx^{2\nu+1}$ (*or* $-x \pm x^{\nu+1} + cx^{2\nu+1}$)

where c is a real number, and ν is a positive (or positive even) integer.

Corollary. *If T is an orientation preserving analytic local diffeomorphism on R, then, given any $r < \infty$, there exists a C^r diffeomorphism σ_1 on R such that $T_1 = \sigma_1^{-1} T \sigma_1$ is in one of the following two normal forms*:

(i′) $T_1(x) = \lambda x, \lambda > 0$;

(ii) $T_1(x) = x + (\pm x)^{\nu+1} + cx^{2\nu+1}, \nu > 0$.

It is known that if $T(x) = \lambda x + o(|x|)$, $|\lambda| \neq 1$, then T can be brought to the normal form (i). See [6] and, for earlier results on analytic cases, p. 272 [3]. It remains to establish the normal form (ii). The proof is divided into two steps, namely Theorem 1 and Theorems 2, 2′, in the sequel.

The technique of estimates in the proof of Theorem 1 is a modification of that of Sternberg's [6] and is also influenced by Hartman's work [3]. The method used in this paper can be generalized and applied to other situations. See [2].

We shall write $D = d/dx$ and, for any real valued function f defined on $U \subset R$,

$$\|f\|_U = \sup \{|f(x)| : x \epsilon U\}.$$

Theorem 1. *Let l be a positive number and r, ν positive integers such that $l > \nu + 1$ and $r > l + \nu$. Let T and T_1 be C^{r+1} local diffeomorphisms on R such that*

$$T(x) = x + h_\nu x^{\nu+1} + o(|x|^{\nu+1}), \qquad h_\nu \neq 0,$$

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