INTERPOLATION IN W^*-ALGEBRAS

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Introduction. This paper continues the author's effort [1], [2] to extend known results about abelian C^*-algebras to the non-abelian case. The main result is Theorem 2.3 which states that if A is a W^*-algebra, A_0 a norm closed two-sided ideal containing a family $\{p_\alpha\}$ of orthogonal projections such that $||a - (\sum_{\alpha \varepsilon K} p_\alpha)a(\sum_{\alpha \varepsilon K} p_\alpha)|| \rightarrow 0$ for all $a \varepsilon A_0$ as K runs over the finite subsets of I directed by inclusion, X a Banach space, and $T : X \rightarrow A$ a bounded linear map with $T(X) + A_0 = A$; then there exists a finite set $K \subset I$ such that if $q = \sum_{\alpha \notin K} p_\alpha$, then $qAq = qT(X)q$.

This result extends (and actually improves) the prototype result of Bade [3] in which $A = l_\infty$ and $A_0 = c_0$. The technique of proof is based on [3]. In the last section we explain how some other results of [3] can be improved. Several corollaries and related results are included in sections three and four.

In §5 we characterize approximate identities in a C^*-algebra to give an alternate formulation of the main result. This characterization also is the exact analog of well-known results about continuous functions (the abelian case).

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1. Preliminaries. Let A be a W^*-algebra and $A_0 \subset A$ a norm closed, two-sided ideal. Define $A_1^* = A_0^\diamond = \{f \varepsilon A^* : f(a) = 0 \text{ for all } a \varepsilon A_0\}$. Since A_0 is an ideal, A_1^* is invariant (i.e. if $f \varepsilon A_1^*$ and $a \varepsilon A$, then the functional $R_af[L_af]$ defined by $R_af(b) = f(ba)[L_af(b) = f(ab)]$ also lies in A_1^*. Thus as in Theorem 1 of [12], $A_1^{*\diamond}$ is a $\sigma(A^{**}, A^*)$ closed ideal of the W^*-algebra [13; 1.73] A^{**}. By Theorem 1.3 of [13, p. 2.3] there exists a central projection $z \varepsilon A^{**}$ such that $A_1^{*\diamond} = zA^{**}$. Further $A^{**} = zA^{**} \oplus (1 - z)A^{**}$, so $A^* = A_1^* \oplus A_0^*$, where $A_0^* = (1 - z)A^{**\diamond}$. Both A_1^* and A_2^* satisfy the following conditions by [13; 1.74] and general duality theory:

1.1 A_0^* and A_1^* are invariant and closed.
1.2 $A_1^* = A_0^\diamond$. i.e. $f \varepsilon A_1^*$, $a \varepsilon A_0$ implies $f(a) = 0$.
1.3 A_0^* is (isometrically isomorphic to) the Banach space dual of A_0 .
1.4 A_0^* and A_1^* are each generated by their positive elements as follows: Each $f \varepsilon A_0^*$ has a unique decomposition $f = f_1 + if_2$ where f_1 and f_2 are self-adjoint. $g \varepsilon A_0^*$ is self-adjoint, there exist unique positive functionals g_1 and g_2 such that $g = g_1 - g_2$ and $||g|| = ||g_1|| + ||g_2||$. Similarly for any invariant closed subspace of A^*.

Since $A^* = A_0^* \oplus A_1^*$, for each $f \varepsilon A^*$ there exist uniquely $f^0 \varepsilon A_0^*$ and $f^1 \varepsilon A_1^*$

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