ON BEST APPROXIMATIONS OF FUNCTIONS OF TWO VARIABLES

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1. Introduction. The subject of this paper is best approximations to continuous functions of two variables with functions of one variable. Specifically, consider the linear space $\mathcal{C}_2$ of real valued continuous functions defined on the unit square in the real Euclidean plane; let $\mathcal{C}$ be the corresponding space of functions with domain $E = [0, 1]$; mark by $\mathcal{S}$ the subspace of $\mathcal{C}$ of function $a(x) + b(y)$. Best approximations to members $f \in \mathcal{C}_2$ are relative to the distance function

$$\mu(f) = \inf_{g \in \mathcal{S}} ||f - g||. \tag{1.1}$$

Our objective is to present a new method for evaluating $\mu(f)$ for each $f \in \mathcal{C}_2$, and for finding the corresponding members of $\mathcal{S}$ which yield this approximation, the main innovation being that the process by which this is accomplished involves only functions of either x or y. A different and beautiful treatment of these problems is contained in a paper of Diliberto and Straus [1]; relevant material may be found also in [2].

2. Statement of the results. Imagine a family of functions, $\mathfrak{g} = \{g_\nu\}$, $g \in \mathcal{C}$, containing at least two non-parallel members; we assume that the index ν varies over some set A of real numbers. The oscillation of a function g_ν is

$$\omega(g_\nu \mid E) = \max_E g_\nu - \min_E g_\nu ;$$

for each function $a \in \mathcal{C}$ we introduce the family

$$\mathfrak{g} - a = \{g_\nu - a : g_\nu \in \mathfrak{g}\}$$

and define the oscillation of $\mathfrak{g} - a$ to be

$$\omega(\mathfrak{g} - a \mid E) = \sup_{\nu \in A} \omega(g_\nu - a \mid E). \tag{2.2}$$

The number

$$\omega(\mathfrak{g}) = \inf_{a \in \mathcal{C}} \omega(\mathfrak{g} - a \mid E) \tag{2.3}$$

is called the least oscillation of the family $\mathfrak{g}$; a function $\alpha \in \mathcal{C}$ is a best ω-approximation to $\mathfrak{g}$ if $\omega(\mathfrak{g}) = \omega(\mathfrak{g} - \alpha \mid E)$.

With each function $f \in \mathcal{C}_2$ we associate the family $\mathfrak{f}$ whose members are defined

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