IDEAL CENTER OF A C^*-ALGEBRA

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Let A be a C^*-algebra, and X its structure space. For every topological space, we denote by $\mathcal{C}^b(T)$ the C^*-algebra of all complex bounded continuous functions on T. If A has a unit, $\mathcal{C}^b(X)$ is canonically isomorphic with the center of A ([1], III, §5). If A has no unit, this is no more true, and the situation is less clear: let $\tilde{A}$ be the C^*-algebra deduced from A by adjunction of a unit, and let $\tilde{X}$ be the structure space of $\tilde{A}$; then X can be canonically identified with the complement in $\tilde{X}$ of a closed point ω; but, as observed in [1], there is no simple relation between $\mathcal{C}^b(\tilde{X})$ and $\mathcal{C}^b(X)$; for example, ω may be in the closure of every point in X (and in this case $\mathcal{C}^b(\tilde{X})$ is reduced to the scalars) with $\mathcal{C}^b(X)$ very large.

In this note, we shall prove that this difficulty can be circumvented by considering an extension A' of A different from $\tilde{A}$. This C^*-algebra A' could be defined in an abstract way, but we prefer to realize it as a C^*-subalgebra of the enveloping von Neumann algebra of A. An essential tool is the following recent result of Dauns and Hofmann ([1], III, 3.5, 3.9 and III, §5): let $y \in A$ and $h \in \mathcal{C}^b(X)$; there exists an element y' in A such that $y' \bmod. I = h(I)\ (y \bmod. I)$ for every $I \in X$.

Notation. For every C^*-algebra A, we denote by $E(A)$ (resp. $P(A)$) the set of all states (resp. pure states) of A, with the weak topology defined by A. We denote by $\hat{A}$ the set of classes of non-zero irreducible representations of A, and by Prim (A) the structure space of A, both with the Jacobson topology. Every $\varphi \in E(A)$ defines canonically a Hilbert space H_φ, a unit vector ξ_φ of H_φ, and a representation π_φ of A in H_φ. Let B be the enveloping von Neumann algebra of A; since B can be canonically identified with the bidual of A, every continuous linear form on A has a canonical extension to B; so, if for example $\varphi \in E(A)$, we shall use the notation $\varphi(x)$ not only for elements x of A, but also for elements x of B. Every representation of A has a canonical extension to B which will be denoted by the same letter. If H is a Hilbert space, we denote by $\mathcal{L}\mathcal{C}(H)$ the C^*-algebra of all compact operators in H; a C^*-algebra isomorphic with $\mathcal{L}\mathcal{C}(H)$ for some H is said to be elementary. If T is a topological space, a point $x \in T$ is said to be separated if, for every point $x' \in T$ not in the closure of x, x and x' have disjoint neighborhoods.

Concerning all these notions, cf. for example [2].

1. Lemma. *Let A be a C^*-algebra, and B its enveloping von Neumann algebra. Let M be the set of all $x \in B$ such that $xA \subset A$, and N the set of all $y \in B$ such that the function $\varphi \mapsto \varphi(y)$ on $E(A)$ is continuous. Then $M + M^* \subset N$.*

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