OPERATORS IN FUNCTION SPACES WHICH COMMUTE WITH MULTIPLICATIONS

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The questions considered here are of the following form. Let $A : L^p(S, \mu) \to L^q(S, \mu)$ be a bounded linear operator, and suppose that it commutes with multiplication by each of a countable family of functions $F \subseteq L^\infty(S, \mu)$. Then what else does A commute with and how can it be characterized? Answer: it commutes with any bounded Baire function of the functions F, and conversely if $g \in L^\infty(S, \mu)$ commutes with all the operators A which commute with all $f \in F$, then g is a bounded Baire function of the functions F. Moreover, F determines a direct integral decomposition of each $L^r(S, \mu)$ over a compact set Λ, and A commutes with the $f \in F$ if and only if it is a corresponding direct integral of operators A_λ.

Such questions have been treated in the Hilbert space case, $p = q = 2$, but in various guises. F. Riesz [8] showed that an operator in a separable Hilbert space which commutes with every operator commuting with a self-adjoint operator B is a function of B. A conjecture of von Neumann proved by Fuglede [2] states that an operator commuting with a normal operator B also commutes with B^*. A lemma useful in the theory of group representations is that an operator in $L^2(G)$ commuting with multiplication by characters of the locally compact abelian group G is necessarily multiplication by an L^∞-function [5]. A theorem in the reduction theory of rings of operators states that an operator in a direct integral of separable Hilbert spaces, $\int_S H_s \, dm(s)$, which commutes with multiplication by all $f \in L^\infty(S, m)$ is multiplication by an operator-valued function [7; cf. 6, §41.2.I]. Each of these facts is contained in the present results or an easy extension of them.

The spaces $L^p(S)$, $1 \leq p \leq \infty$, are treated in §1 below. We examine the same questions for $L^\infty(S)$ and $C(S)$ in §2. The extent to which the main results generalize to vector-valued functions is summarized in §3.

1. Operators in Lebesgue spaces. All results and proofs are valid in both the real and complex cases. We shall assume for definiteness that all scalar functions are complex. Let (S, Σ, μ) be a positive measure space. For $1 \leq p \leq \infty$, $L^p(S, \mu) = L^p(S)$ is the usual Lebesgue space, with norm $||g||_p$. We shall generally speak of the elements of $L^p(S)$ as "functions," and a statement such as "g vanishes on B" is to be taken to mean "g vanishes a.e. on B".

Multiplication by $f \in L^\infty(S)$ is a bounded linear operator in $L^p(S)$, all p. If $A : L^p(S) \to L^q(S)$ and $A(fg) = f(Ag)$, all $g \in L^p(S)$, we say A *commutes with* f.

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