ON THE LOGARITHMIC CAPACITY AND CONFORMAL MAPPING

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1. Results. Let cap E denote the logarithmic capacity of the compact plane set E.

Theorem 1. *Let $f(z) = z + a_0 + a_1 z^{-1} + \cdots$ be univalent in $|z| > 1$. If A is a closed set on the unit circle $|z| = 1$ then*

$$\text{cap } f(A) \geq (\text{cap } A)^2, \tag{1.1}$$

and this inequality is best possible.

Here we define $f(A)$ as the set of all limit points of $f(\zeta)$ as $\zeta \to z$, $z \in A$. The extremal function is not essentially unique.

This inequality was proved by M. Schiffer [9; 432] for the case that A is an arc. His proof uses a variational method and does not carry over to the general case. Our proof will be based on an inequality of Golusin.

Theorem 2. *Let E_1 and E_2 be compact plane sets such that $E_1 \cup E_2$ is connected. Then*

$$\text{cap } (E_1 \cup E_2) \leq \text{cap } E_1 + \text{cap } E_2 . \tag{1.2}$$

This inequality was proved by M. Schiffer [8], [9] under the further assumptions that the sets E_1 and E_2 are connected and that there is a closed Jordan curve J such that E_1 lies in the interior of J and E_2 in the exterior, except for the points $E_1 \cap E_2$ which lie on J. Obviously, (1.2) need not hold if $E_1 \cup E_2$ is not connected.

Corollary 1. *Let E_k ($k = 1, 2, \cdots$) be continua such that*

$$E = \bigcup_k E_k$$

is a continuum. Then

$$\text{cap } E \leq \sum_k \text{cap } E_k . \tag{1.3}$$

Corollary 2. *If E is a continuum of linear measure $l(E)$, then*

$$l(E) \geq 4 \text{ cap } E, \tag{1.4}$$

and this inequality is best possible.

This estimate was proved by M. Fekete [4] for the case that E is an arc.

2. The distortion of the logarithmic capacity. If F is a plane compactum and

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