FRACTIONAL INTEGRALS AND HANKEL TRANSFORMS

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1. Introduction. The problem of determining all pairs of functions $f(t)$ and $g(t)$ in $L^2(0, \infty)$ such that, for some definite Hankel transformation

(i) $g(t)$ is the transform of $f(t)$,
(ii) both $f(t)$ and $g(t)$ vanish throughout a certain neighborhood of $t = 0$,

was formulated and partially solved by de Branges [1]. Using differential operators, he found a complete solution provided the Hankel transform is of order $\nu = 0$. Rovnyak [8] then gave another solution by means of unitary transformation on $L_+^2(-\infty, \infty)$.

De Branges conjectured that similar results must hold for transforms of arbitrary order ν. Here we carry out such an extension using Rovnyak's technique when $\nu > 0$. The results are summed up in two theorems.

2. Notation and basic definitions. We consider only those functions which belong to $L^2(0, \infty)$ and interpret the convergence of the integrals in the metric of L^2. Two functions are treated as identical if they differ only on a set of measure zero.

We use Tricomi's form of Hankel transform. Thus $g_1(t)$ is the Hankel transform of order ν of $f(t)$ if

$$g_1(t) = \mathfrak{H}_\nu\{f(u); t\} = \int_0^\infty f(u) J_\nu[2\sqrt{(ut)}]\, du.$$

This is different from the form used by de Branges and Rovnyak. However, a change from one form of Hankel transform into the other presents no difficulty especially when the functions are square integrable.

A_ν denotes the set of those functions $f(t)$ in $L^2(0, \infty)$ for which the function $g(t) = t^{\nu/2}\mathfrak{H}_\nu\{u^{-\nu/2}f(u); t\}$, $\nu > 0$, also is in $L^2(0, \infty)$.

We observe that $f(t)$ is in A_ν if and only if $g(t)$ is so. From now on, we shall assume $f(t)$ and $g(t)$ related as above unless specified otherwise.

$L_2^{(\lambda)}$ is a subset of $L^2(0, \infty)$ defined by Erdélyi [2] as follows: If $f(t)$ is in $L^2(0, \infty)$ and its Mellin transform $f_1(\tau)$ is such that $\tau^{-\lambda} f_1(\tau)$ is in $L^2(-\infty, \infty)$, then $f(t)$ is in $L_2^{(\lambda<0)}$.

$L_+^2(-\infty, \infty)$ is the space of functions $F(z)$, $z = x + iy$, which are analytic in $y > 0$ and satisfy the condition

$$\sup_{y>0} \left\{\int_{-\infty}^\infty |F(x + iy)|^2\, dx\right\} < \infty.$$

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