INSERTION OF OPEN FUNCTIONS

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1. Introduction. In [5] Max Weiss and the first author gave a sufficient condition for the insertion of a Darboux function between two comparable real-valued functions of a real variable. It was shown that this condition is met when both functions are Darboux and in the first class of Baire. Moreover, an example was given of two comparable Darboux functions belonging to the second class of Baire and admitting no Darboux function between them.

In this paper we extend the above result to apply to functions from R^n into R, and simultaneously strengthen it so that the inserted function is both Darboux and open. In particular, we show that the hypothesis of the strengthened version is satisfied when (1) both functions are Darboux and in the first class of Baire, and when (2) the upper function is lower semi-continuous and the lower function is upper semi-continuous.

We also weaken the requirement that the inserted function be Darboux and open to that of being open, and derive a reasonable sufficient condition for the insertion of an open function between two comparable functions. In particular, this condition is satisfied when (1) both functions are Darboux or when (2) one is continuous and the other open.

2. Definitions and terminology. Suppose f and g are real-valued functions defined on some set A. We say that $g < f$ on A if $g(x) < f(x)$ for each $x \in A$. We say that a real-valued function h defined on A is *inserted between* g and f on A if $g < h < f$ on A. In the sequel R will denote the set of real numbers and I will denote an arbitrary real interval.

The cardinality of a set A is denoted by $|A|$, and we denote $|R|$ by $\mathbf{c}$. By a cardinal number we mean an ordinal which is not equipollent with a smaller ordinal.

A function from I into R is said to be *Darboux* provided it maps intervals onto intervals (that is, connected sets onto connected sets). The notion of a Darboux function has been extended to more general domains in many ways. The usual and most fruitful way is to define a Darboux function relative to a base $\mathfrak{U}$ of the domain space as follows [1], [6]: Let f be a function from the topological space X into R, and let $\mathfrak{U}$ be a base for X. If for each

$$U \in \mathfrak{U}, \quad x, y \in \bar{U}, \quad \text{and} \quad \lambda \in (f(x), f(y))$$

there exists $z \in U$ such that $f(z) = \lambda$, then f is said to be *Darboux relative to* $\mathfrak{U}$. This agrees with the usual definition of a Darboux function when the domain is an interval.

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