ON COMMUTATIVE LOCALLY M-CONVEX ALGEBRAS

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If A is a commutative Banach algebra, M is its spectrum (non-zero continuous $\mathbf{C}$-valued homomorphisms, Gelfand topology), and M^+ is its extended spectrum (all continuous $\mathbf{C}$-valued homomorphisms, Gelfand topology), then M^+ is the one-point compactification of M, where the point at infinity, m_0, is the zero homomorphism on A. In particular, if we know $M(A)$, then we know $M^+(A)$. We show that this is not the case for locally m-convex algebras, not even for F-algebras. Specifically, we exhibit several F-algebras A_i such that $M(A_i) = T$ for all i and such that no two $M^+(A_i)$ are homeomorphic, although in each case we have added only one point to T.

We give a general method for topologically adding a point t_0 to a Hausdorff space T in terms of ideals in the lattice of closed subsets of T, the correspondence between ideals and topologies on $T \cup \{t_0\}$ whose restriction to T is the given topology being one-to-one and order-preserving. If A is a locally m-convex algebra and M is its spectrum, then the closed A-bounded subsets of M form an ideal and M^+ (the extended spectrum) is the one-point extension determined by this ideal.

If A is a commutative Banach algebra with identity, then M is compact and m_0 is an isolated point of M^+. If A is semi-simple, the converse is also true so we have a topological condition on M implying an algebraic statement about A. We consider the question of the existence of an identity for locally m-convex algebras and obtain several conditions involving A and related Banach algebras equivalent to the statement that m_0 is isolated. In the presence of semi-simplicity these imply that A has an identity. However, the statement that m_0 is isolated is not a topological statement about M. In general there is no topological property of M which guarantees that A will have an identity. To demonstrate this we exhibit two F-algebras, one with an identity and one without, that have the same spectrum. In fact, one algebra is obtained by adjoining an identity to the other one.

Lastly, we consider the closed ideals of an F-algebra A and show that the closed A-bounded subsets of M play the same role in our setting as the compact sets play in the study of ideal theory for Banach algebras. For example, if $S \subseteq M$ is a hull, then $k(S)$ is modular if, and only if, S is A-bounded. A key theorem in the development is the following. If I and J are closed ideals in A with disjoint hulls and if I is modular, then $I + J = A$. This is a trivial theorem for Banach algebras (since M is the space of all maximal ideals) and false for locally m-convex algebras in general. As a corollary to this theorem we obtain

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