ON THE EXISTENCE OF CERTAIN DISJOINT ARCS IN GRAPHS

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1. Introduction. When is it possible for a communications network to support simultaneously any set of n two-way conversations (involving $2n$ distinct parties), the lines being routed in such a way that no two of them intersect? Stated in terms of the theory of graphs the question becomes one of determining for each integer $n \geq 2$ when a finite undirected graph G with vertex set $V(G)$ satisfies the condition:

$\mathfrak{Q}_n : |V(G)| \geq 2n$, and given $2n$ distinct vertices $a_1, \cdots, a_n, b_1, \cdots, b_n \in V(G)$, there exist n disjoint arcs $P_i[a_i, b_i]$, $(i = 1, \cdots, n)$, in G.

It is the purpose of this article to present some necessary conditions and some sufficient conditions for G to satisfy $\mathfrak{Q}_n$. The special cases where $n = 2$ and $n = 3$ are investigated somewhat more fully.

2. Background and prerequisites. With minor exceptions, the language of this paper is that of O. Ore [8]. G will always denote a finite, undirected, connected graph without loops or multiple edges. Its vertex set will be denoted by $V(G)$, or simply by V.

The complete graph on m vertices will be denoted by K_m. The vertex-connectivity of G, denoted by $\lambda = \lambda(G)$, is defined as follows: $\lambda(K_m) = m - 1$ for $m \geq 2$; otherwise $\lambda(G)$ is the number of vertices in a smallest separating set of G.

A family of arcs in G is said to be *openly disjoint* if the arcs in the family are pairwise disjoint except at common endpoints, if any. If X and Y are disjoint subsets of $V(G)$, an *XY-arc* P has one end-point in each of X and Y and contains no other vertices in $X \cup Y$.

For each positive integer n, consider the condition

$\mathfrak{D}_n$: If X and Y are disjoint, non-empty subsets of $V(G)$, and μ and υ are functions from X and Y, respectively, into the positive integers such that

$$\sum_{x \in X} \mu(x) = n = \sum_{y \in Y} \upsilon(y),$$

then G contains an openly disjoint family of n XY-arcs such that each vertex $x \in X$ is an end-point of $\mu(x)$ of these arcs and each vertex $y \in Y$ is an end-point of $\upsilon(y)$ of these arcs.

This formulation of $\mathfrak{D}_n$ is due to G. A. Dirac [6]. The same reference contains the following result:

Theorem 1. *If $\lambda(G) \geq n$, then G satisfies $\mathfrak{D}_n$.*

Received February 3, 1967; in revised form January 24, 1968.