CONTINUOUS DEPENDENCE OF BOUNDED SOLUTIONS OF A LINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION UPON INTERIOR CAUCHY DATA

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1. Introduction. In this paper we consider the solution of a linear second order parabolic differential equation in a rectangle

$$\Gamma = \{(x, t) \mid 0 < x < 1, 0 < t \leq T\},$$

where the usual initial data is replaced by Cauchy data, u and u_x, specified on a vertical line segment inside the rectangle. The problem is not well posed in the sense of Hadamard. As an example, consider the solutions

$$u(x, t) = n \exp\{-4n^2\pi^2 t\} \sin 2n\pi x$$

of

$$\begin{aligned} u_{xx} - u_t &= 0, \qquad \text{in} \quad \Gamma, \\ u(0, t) &= u(1, t) = 0, \qquad 0 < t \leq T, \\ u(\tfrac{1}{2}, t) &= 0, \qquad \tfrac{1}{2}T \leq t \leq T, \\ u_x(\tfrac{1}{2}, t) &= 2n^2(-1)^n\pi \exp\{-4n^2\pi^2 t\}, \qquad \tfrac{1}{2}T \leq t \leq T. \end{aligned}$$

Obviously, as n tends to infinity, the data sequence tends to zero in the maximum norm while the solution sequence tends to infinity in the maximum norm. Thus, such problems do not depend continuously upon the data without some additional information. Physically, it is natural to assume that solutions to such problems are bounded in absolute value by a known positive constant. With this additional specification of data, we demonstrate that on compact subsets of Γ the solution depends Hölder continuously upon the data.

For problems which are not well posed in the sense of Hadamard that are subject to additional data such as the specification of an a priori bound for the solution, it is usually easy to show via equicontinuity arguments that on compact subsets of the domain of definition of the solution, the solution depends continuously upon the data. However, in order to estimate the effectiveness of numerical procedures for approximating these solutions, it is important to know the form of the continuous dependence on the data. A large body of recent work has been concerned with the estimation of the form of the continuous dependence on the data for such problems as the Cauchy problem for Laplace's equation [24], [30], analytic continuation [8], [15], continuation of solutions of

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