TOPOLOGIES ON QUOTIENT FIELDS

By Ellen Correl

Let A be a principal ideal domain, P a representative system of prime elements of A, K the quotient field of A. Our purpose is to prove that if $(K, \mathfrak{J})$ is a Hausdorff indiscrete (i.e., not discrete) topological field for which the open A-submodules of K form a fundamental system of neighborhoods of zero, then $\mathfrak{J}$ is the supremum of a family of p-adic toplogies.

For each nonzero $x \varepsilon K$ there exist a unique family $(v_p(x))_{p \varepsilon P}$ of integers, all but finitely many of which are zero, and a unique unit u of A such that

$$x = u \prod_{p \varepsilon P} p^{v_p(x)}.$$

For each $p \varepsilon P$, the function v_p (defined also at zero by $v_p(0) = +\infty$) is, of course, the familiar p-adic valuation on K, and if

$$V_{p,n} = \{x \varepsilon K: v_p(x) \geq n\},$$

then $(V_{p,n})_{n \geq 0}$ is a fundamental system of neighborhoods of zero for the p-adic topology $\mathfrak{J}_p$ on K; this topology is the topology defined by the nonarchimedean absolute value $| \ |_p$, where $|x|_p = 2^{-v_p(x)}$. Clearly each $V_{p,n}$ is an open A-submodule of K for $\mathfrak{J}_p$. Hence if $\mathfrak{J}$ is the supremum of a family of p-adic topologies on K, then $(K, \mathfrak{J})$ is an indiscrete Hausdorff topological field for which the open A-submodules form a fundamental system of neighborhoods of zero.

1. Submodules of the quotient field. Throughout, G is a nonzero submodule of the A-module K. Consequently $G \cap A$ is a nonzero ideal of A, and therefore there is a unique generator a_G of the ideal $G \cap A$ that is the product of elements of P.

Lemma 1. *If $a, b, c \varepsilon A$, if $a \varepsilon G$ and $abc^{-1} \varepsilon G$, and if b and c are relatively prime, then $ac^{-1} \varepsilon G$.*

Proof. There exist $x, y \varepsilon A$ such that $xb + yc = 1$. Hence

$$ac^{-1} = (xb + yc)ac^{-1} = xabc^{-1} + ya$$

belongs to G.

Lemma 2. *If $a, b, c \varepsilon A$, if $ab^{-1} \varepsilon G$ and $ac^{-1} \varepsilon G$, and if b and c are relatively prime, then $a(bc)^{-1} \varepsilon G$.*

Received January 30, 1967. The results presented here, for the special case of the rational field, are contained in the author's doctoral dissertation (Purdue, January, 1958), written under the supervision of Merrill E. Shanks. Research for the dissertation was supported by the Purdue Research Foundation and the United States Air Force.