MAXIMAL SEMI-ALGEBRAS OF NON-NEGATIVE FUNCTIONS ON A LOCALLY COMPACT SPACE

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The purpose of this note is to settle a question posed in [2]: what are the maximal uniformly closed subsemi-algebras of $C_0^+(E)$, where E is a locally compact Hausdorff space, $C_0^+(E)$ is the set of all continuous non-negative functions defined on E and vanishing at infinity? (A semi-algebra is a wedge closed under pointwise multiplication; a subsemi-algebra M of $C_0^+(E)$ is maximal closed iff no proper closed subsemi-algebra of $C_0^+(E)$ properly contains M.)

This question was answered in [1] in the case that E is compact. For this case, the fact that the continuous non-negative functions formed a set with non-void interior in the uniform topology was very important. For the general case, this property is not available, and one is forced either to throw the burden of the proof back onto the compact case or else to try a constructive technique. That each maximal closed semi-algebra is of a certain form (viz. a geometric semi-algebra) is a consequence (via the one-point compactification) of known results for the compact case. That each geometric semi-algebra is maximal can either be shown directly or deduced from the compact case. I am indebted to the referee for showing how the latter proof would proceed. Either proof requires a lemma (Lemma 1) asserting the existence of a non-negative unbounded continuous function which grows sufficiently slowly that it is integrable with respect to a prescribed probability measure.

1. Statement of the theorem and part of the proof. Let σ be a Radon probability measure on E. For f, a bounded non-negative continuous function on E, the *geometric mean of f with weight σ*, denoted by $GM_\sigma f$, is defined by

$$GM_\sigma f \equiv \begin{cases} \exp \int \log f \, d\sigma & \text{if } \log f \text{ is } \sigma\text{-integrable} \\ 0 & \text{if } \log f \text{ is not } \sigma\text{-integrable.} \end{cases}$$

A *geometric semi-algebra* is of the form

$$H_{\sigma,\xi} \equiv \{f : f \in C_0^+(E),\ f(\xi) \leq GM_\sigma f\}$$

where ξ is a point of E and σ is a probability measure having no mass at ξ. That $H_{\sigma,\xi}$ is a closed semi-algebra is straightforward to verify; see [1, Theorem 1.4].

Received January 23, 1967; in revised form March 20, 1967. The author was an NATO Postdoctoral Fellow at Yale University during the academic year 1966–1967.