# THE BURKILL-CESARI INTEGRAL

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**Introduction.** In two recent papers [3], [4], Cesari introduced by means of axioms the concept of a quasi-additive vector-valued set function $\phi(I) = (\phi_1, \cdots, \phi_k)$ in a class $\{I\}$ of sets $I$ of a space $A$, a concept of a mesh $\delta(D)$ of certain finite collections $D$ of sets $I$, and the notion of an integral $\int \phi$ of such a function $\phi$ defined by a standard process of limit as $\delta(D) \rightarrow 0$. This process is more general than usual since $\delta$ is not required to decrease nor approach zero by refinements only. Although the original motivation for the concept of quasi-additivity is to be found in surface area theory, Cesari showed in [3] that this same property is possessed by interval functions which are encountered in many other problems (Cauchy integrals, Lebesgue–Stieltjes integrals, Jordan length). Moreover, it was shown that this formulation includes Weierstrass-type integrals $\int f(\mathcal{V}, \phi)$ over a Euclidean variety $\mathcal{V}$ with a quasi-additive set function $\phi$ of bounded variation, and thus in particular the Weierstrass integrals of the calculus of variations for curves and surfaces. Under suitable conditions, Cesari proved that the integral $\int f(\mathcal{V}, \phi)$ admits a representation as a Lebesgue–Stieltjes integral with respect to a measure induced by $\phi$ [4].

The main purpose of this paper is to develop the basic notions of quasi-additivity and integral in a general and abstract setting. As consequences of this development, the range of applicability of these concepts is substantially increased. Thus, for example, we are able to show that many of the integration processes defined by refinement of partitions rather than by a mesh $\delta$ may now be formulated within the context of the general theory of integration of quasi-additive set functions (which, incidentally, cannot be done using the machinery in [3]). In addition, as will be seen in another paper, where the generalized Weierstrass-type integrals $\int f(\mathcal{V}, \phi)$ will be studied in detail, this setting will form a natural background for the formulation of abstract convergence and semi-continuity theorems.

The contents of this paper are as follows. In §1, the definition of the integral is examined. Instead of requesting that the set function $\phi$ should have its range in a Euclidean space, we now assume that the value of $\phi$ lie in some Hausdorff locally convex topological vector space $E$. The notion of a mesh function $\delta$ is abstracted and the limiting process used to define the integral

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