SYMMETRY IN COMPLEX INVOLUTORY BANACH ALGEBRAS

By Satish Shirali

Kaplansky [2] defines a C-symmetric algebra as a complex involutory Banach algebra in which every closed commutative involutory subalgebra is symmetric. He then asks whether a C-symmetric algebra is actually symmetric. It is a consequence of Theorem 4.3 of [2] that in a C-symmetric algebra every hermitean element has a real spectrum. Kaplansky notes that he assumes the involution to be only real linear and not necessarily conjugate linear or continuous. If the latter assumptions are made, then the converse of the above result is also known: If every hermitean element has a real spectrum, then the algebra is C-symmetric. The purpose of the present paper is to show that under these assumptions a C-symmetric algebra is symmetric.

The definitions and notations to be introduced here will be used throughout this paper except where specified.

A is a complex Banach algebra with a norm-preserving involution *, and an identity e. H is the real Banach subspace of hermitean elements; C [resp. D] is the closed cone generated by elements of the form x^*x [resp. $x^*x + xx^*$] and is called the *natural cone* [resp. *quasicone*] of A; a linear functional f on A is said to be *positive* [resp. *quasipositive*] if f takes non-negative values on C [resp. D]; the set of all positive [resp. quasipositive] functionals f such that $f(e) = 1$ will be denoted by F [resp. G].

For $x \in A$, sp (x) is the spectrum of x; $\sigma(x)$ is the smallest closed interval containing the real numbers in sp (x); $||x||_{sp}$ is the spectral radius of x; finally $F(x)$ [resp. $G(x)$] is the set of all $f(x)$ where $f \in F$, [resp. $f \in G$].

The **radical* is the *ideal of all elements vanishing under every $f \in F$; Naimark [5] calls it the *reducing ideal*. **Semisimple* (or *reduced* in the terminology of [5]) will mean *radical $= \{0\}$.

A is said to be C-symmetric if every $h \in H$ has a real spectrum; and symmetric if for every $x \in A$, $e + x^*x$ has an inverse in A. Analogously, A is said to be quasisymmetric if for every $x \in A$, $e + x^*x + xx^*$ has an inverse in A.

If the involution is merely continuous, then there is an equivalent norm which is preserved by the involution. Therefore no generality is lost by assuming $||x^*|| = ||x||$. The assumption of the identity will be removed at the very end. We shall need to assume *semisimplicity in Lemma 4; this restriction is also removed later, so that there is no harm in assuming it now.

Methods used to study F apply with trivial changes to G. First we note that $F \subset G$. We can show that every $g \in G$ is continuous on H with norm $g(e) = 1$. Just like F, G is also a weak* compact convex subset of the conjugate space of

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