A SEQUENCE OF NUMBERS RELATED TO THE EXPONENTIAL FUNCTION

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1. Introduction. The Bernoulli numbers B_0, B_1, B_2, ... are defined by

$$\frac{x}{e^x - 1} = \sum_{s=0}^{\infty} B_s \frac{x^s}{s!}.$$

Properties of these numbers are well known [4]. Other generating functions of the type

$$\frac{x^n}{n!}\left(e^x - \sum_{r=0}^{n-1} \frac{x^r}{r!}\right)^{-1} = \frac{x^n}{n!}\left(\sum_{r=n}^{\infty} \frac{x^r}{r!}\right)^{-1} \tag{1.1}$$

have not been studied, although van der Pol in [5; 235] briefly discusses the numbers β_0, β_1, β_2, $\cdots$ defined by

$$\frac{x^3}{3!}\left(\sum_{r=3}^{\infty} \frac{(r-2)x^r}{r!}\right)^{-1} = \sum_{s=0}^{\infty} \beta_s \frac{x^s}{s!}.$$

When $n = 1$ in (1.1), we have the Bernoulli case. In this paper we shall examine the case $n = 2$.

We first define a polynomial $A_n(z)$ which is analogous to the Bernoulli polynomial. Put

$$\frac{x^2}{2} \frac{e^{xz}}{e^x - x - 1} = \sum_{s=0}^{\infty} A_s(z) \frac{x^s}{s!}. \tag{1.2}$$

If we define $A_s(0) = A_s$ $(s = 0, 1, \cdots)$, it follows from (1.2) that

$$A_n(z) = \sum_{s=0}^{n} \binom{n}{s} A_s z^{n-s}. \tag{1.3}$$

Since

$$\frac{\frac{x^2}{2}}{e^x - x - 1} = \sum_{s=0}^{\infty} A_s \frac{x^s}{s!}, \tag{1.4}$$

we have $A_0 = 1$, and for n > 0

$$\sum_{s=0}^{n} \binom{n+2}{s} A_s = 0. \tag{1.5}$$

Using (1.5), we can easily compute the first few values of A_n. We have $A_0 = 1$, $A_1 = -\frac{1}{3}$, $A_2 = \frac{1}{18}$, $A_3 = \frac{1}{90}$, $A_4 = -\frac{1}{270}$, $A_5 = -\frac{5}{1134}$. A more extensive table of values is given at the end of the paper.

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