DOUBLY STOCHASTIC ASSOCIATED MATRICES

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Let A be an n-square complex matrix. A matrix $K(A)$ whose entries are polynomials in the entries of A and which satisfies

$$K(AB) = K(A)K(B) \tag{1}$$

for all A and B is called an associated matrix [3; 76]. The most familiar instances of associated matrices are $\Pi^r(A) = A \otimes \cdots \otimes A$, the r-th Kronecker power of A, $P_r(A)$, the r-th induced power matrix, and $C_r(A)$, the r-th compound of A. Associated matrices arise in a natural way as matrix representations of linear transformations on symmetry classes of tensors. However, here we shall give a direct matrix definition of a restricted class of associated matrices $K(A)$ in terms of certain generalized submatrices of A.

Let G be a subgroup of order g of the symmetric group S_r and let χ be a character on G of degree 1. If X is an r-square matrix then set

$$d_\chi^G(X) = \sum_{\sigma \varepsilon G} \chi(\sigma) \prod_{i=1}^{r} X_{i\sigma(i)} . \tag{2}$$

The function d_χ^G is called a generalized matrix function [2]. Next, let $\Gamma_{r,n}$ denote the totality of n^r sequences $\omega = (\omega_1, \cdots, \omega_r)$, $1 \leq \omega_i \leq n$ and define an equivalence relation on $\Gamma_{r,n}$ by: $\omega \sim \beta$ if there exists a $\sigma \varepsilon G$ for which $\omega^\sigma = (\omega_{\sigma(1)}, \cdots, \omega_{\sigma(r)}) = \beta$. Let Δ be a system of distinct representatives with respect to this equivalence relation in $\Gamma_{r,n}$ so chosen that each element is first in its equivalence class in lexicographic order. Let $\bar{\Delta}$ be the subset of Δ for which

$$\sum_{\sigma \varepsilon G_\omega} \chi(\sigma) = \nu(\omega) \neq 0$$

where $G_\omega = \{\sigma \varepsilon G | \omega^\sigma = \omega\}$, i.e., G_ω is the stabilizer in G of ω. In other words, $\omega \varepsilon \bar{\Delta}$ if and only if the character χ is identically 1 on G_ω (so that $\nu(\omega)$ is the order of G_ω). For example, if $G = S_r$, $\chi = \epsilon$, then $\bar{\Delta}$ is the set of strictly increasing sequences ω of length r, $1 \leq \omega_1 < \cdots < \omega_r \leq n$ and $\nu(\omega) = 1$. Again, if $G = S_r$ and $\chi \equiv 1$, then $\Delta = \bar{\Delta}$ is the set $G_{r,n}$ of all $\binom{n+r-1}{r}$ non-decreasing sequences of length r chosen from $1, \cdots, n$, and

$$\nu(\omega) = \prod_{t=1}^{n} m_t(\omega)! \tag{3}$$

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