INTERPOLATING FAMILIES AND GENERALIZED CONVEX FUNCTIONS

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1. Introduction. Beckenbach [1] gave the following definition: A family F of real-valued continuous functions $\{\phi(t)\}$ on an interval $a < t < b$ is called a 2-parameter family if, for every pair of distinct numbers t_0, $t_1 \in (a, b)$ and arbitrary real numbers y^0 and y^1, there is a *unique* element $\phi(t) \in F$ satisfying $\phi(t_0) = y^0$ and $\phi(t_1) = y^1$.

The considerations of [4] indicate the desirability of an extension of this notion to functions of $x = (x^1, \cdots, x^n)$ in a subset of R^n. In this extension, we shall require the unique interpolation property at $n + 1$ points $(x_0, \cdots, x_n)$ which are not only distinct but are the vertices of a non-degenerate simplex $S(x_0, \cdots, x_n)$. In other words, it will be required that if $x_j = (x_j^1, \cdots, x_j^n)$, then

$$(1.1) \qquad \det \begin{bmatrix} x_0^1 & \cdots & x_0^n & 1 \\ x_1^1 & \cdots & x_1^n & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n^1 & \cdots & x_n^n & 1 \end{bmatrix} \neq 0.$$

Definition. Let $\Omega \subset R^n$ be a convex open set. A family F of real-valued continuous functions $\{\phi(x)\}$ on Ω will be called an *$(n, n + 1)$-interpolating family* if, for every set of $n + 1$ points $x_0, \cdots, x_n$ of Ω satisfying (1.1) and for every set of $n + 1$ real numbers $y^0, \cdots, y^n$, there exists a *unique* element $\phi \in F$ satisfying $\phi(x_j) = y^j$ for $j = 0, \cdots, n$.

The simplest example of an $(n, n + 1)$-interpolating family is the set of functions

$$(1.2) \qquad \phi(x) = c_0 + c_1x^1 + \cdots + c_nx^n,$$

with $c_0, \cdots, c_n$ arbitrary constants. If $n = 1$, then the notion of an $(n, n + 1)$-interpolating family reduces to that of a 2-parameter family. It will be shown that $(n, n + 1)$-interpolating families have properties analogous to those of 2-parameter families.

The following notation and terminology will be used: $S(x_1, \cdots, x_m)$ is the convex closure of the set of points $x_1, \cdots, x_m$; $p(x_1, \cdots, x_m)$ denotes the smallest flat containing $x_1, \cdots, x_m$ (so that, for example, $S(x_1, \cdots, x_m) \subset p(x_1, \cdots, x_m)$ and (1.1) means that $\dim p(x_0, \cdots, x_n) = n$); π will denote a hyperplane and $\pi^\pm$ the open half-spaces bounded by π; for brevity, π and $\pi^\pm$ will also be used to denote the intersections $\pi \cap \Omega$ and $\pi^\pm \cap \Omega$ of Ω and the hyper-

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