A TOPOLOGICAL PROPERTY OF BING'S DECOMPOSITION OF E^3 INTO POINTS AND TAME ARCS

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1. Introduction. In [3] Bing gave the first example of a point-like upper semi-continuous decomposition G of E^3 such that its decomposition space, E^3/G, is not homeomorphic to E^3. He showed that E^3/G was not E^3 by proving the non-existence of a homeomorphism between these spaces. In [5; 9] and [3; 498] he asked if there were some topological property of E^3/G which would distinguish it from E^3. In [1], Armentrout announced that E^3/G has certain points without small simply connected neighborhoods. In this paper we show (Theorem 2) that each point of $P(A_0)$ has no small neighborhood bounded by a 2-sphere (P is the projection map of E^3 onto E^3/G, and A_0 is the sum of the non-degenerate elements of G). In [6] it was announced that E^3/G has this property, but the announcement was subsequently withdrawn. In the proof we present here to show the non-existence of a small neighborhood bounded by a 2-sphere, essential use is made of Bing's paper [3]. Hence, while we do obtain a topological property of E^3/G which distinguishes it from E^3, we do not obtain an independent method of proving that E^3/G is not E^3. In §6 we show that each point of E^3/G has an arbitrarily small neighborhood bounded by a 2-sphere with one handle (the boundary of a solid torus).

2. Preliminaries. In §2 of [3] Bing uses A, A_i, A_{ij}, A_{ijk}, $\dots$, (each letter in a subscript is one of the integers 1, 2, 3, and 4) to define his decomposition of E^3. Let $B_1 = \sum A_i$, $B_2 = \sum A_{ij}$, $\cdots$, let X denote a component of some B_α (X is a homeomorphic image of A), and let X_1 , X_2 , X_3 , and X_4 denote the components of $B_{\alpha+1}$ contained in X. (See Figure 1.)

Let F and F_i be the centers of X and X_i , respectively. (See §7 of [3].) Note that F, a topological figure eight, consists of an upper loop F_u and a lower loop F_l . Also, in §7 of [3], Bing defines Properties P and Q relative to the fixed disks D_1 and D_2 . We shall use Properties P and Q, but these properties will be used here relative to various pairs of disks E_1 and E_2 . That is, relative to disks E_1 and E_2 in E^3 and a component X of B_α , we have the following definitions.

Property P. A topological figure eight has Property P if it contains two points x and y in opposite loops such that any arc from x to y in it intersects both E_1 and E_2 .

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