SUBMERSIONS AND GEODESICS

By Barrett O'Neill

1. Introduction. For Riemannian manifolds M and B, a *submersion* $\pi : M \to B$ is a mapping onto B which has maximal rank and preserves lengths of horizontal tangent vectors to M [4]. (A tangent vector to M at p is *horizontal* if orthogonal to the *fiber* $\pi^{-1}\pi(p)$ through p, *vertical* if tangent to the fiber.) Submersions occur frequently in Riemannian geometry; many examples are considered in [3], [6], notably the projection $G \to G/H$ of a Riemannian homogeneous space. A submersion $\pi : M \to B$ is described infinitesimally by tensors T and A on M introduced in [6], where the relations between the curvatures of M, B, and the fibers $\pi^{-1}(b)$ are expressed in terms of these tensors.

Our aim here is to compare the geodesics of M and B, and to show how conjugacy and index on a geodesic in B derive from the geometry of M and the fibers. When M is complete, any geodesic segment β in B is the projection $\pi \circ \gamma$ of a horizontal geodesic segment γ in M. Our main result is a formula (Theorem 3) relating the index form on γ, with fibers as endmanifolds, to the fixed endpoint index form on $\beta = \pi \circ \gamma$. As a consequence we can identify conjugacy and index on β with conjugacy and index on γ-with fibers as endmanifolds at either one or both endpoints of γ. When only fixed endpoint conjugacy on γ is known we still obtain information about conjugacy on β (Theorem 5). Roughly speaking, if a conjugate point on γ does not project to a conjugate point on β (order of conjugacy, at least, need not be preserved) it is nevertheless responsible for a conjugate point occurring earlier on β. Finally we look at a special case which suggests how detailed information about M and the tensor A can be used to locate conjugate points in B.

For the curvature relations mentioned above we refer to [6], but we now summarize briefly the basic properties of the tensors T and A. Let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of each tangent space of M onto its (complementary) subspaces of horizontal and vertical vectors. Then for arbitrary vector fields E and F on M, $T_E F = \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F)$. Thus T is one formulation of the second fundamental form of all fibers. For $A_E F$, simply exchange $\mathcal{H}$ and $\mathcal{V}$ in the formula for $T_E F$. Subsequent computations make frequent use of the following properties of T and A: (1) T_E and A_E are, at each point, skew-symmetric linear operators on the tangent spaces of M; each sends horizontal vectors to vertical, and vertical to horizontal; (2) T is vertical, A horizontal, that is, $T_E = T_{\mathcal{V}E}$ and $A_E = A_{\mathcal{H}E}$; (3) for vertical vector fields, $T_V W = T_W V$; for horizontal vector fields, $A_X Y = \frac{1}{2}\mathcal{V}[X, Y] = -A_Y X$. This last fact, proved in [6], shows that A is the integrability tensor of the horizontal distribution on M.

Received June 6, 1966. This work was supported by a National Science Foundation grant.