OPERATOR ALGEBRAS GENERATED BY PROJECTIONS

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The purpose of this note is to prove that certain $*$-algebras of bounded linear operators on a complex Hilbert space are generated (as algebras) by their projections.

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ a $*$-algebra of operators on $\mathcal{H}$, and $\mathfrak{M}$ the algebra of $n \times n$ matrices ($n \geq 2$) with entries from $\mathcal{A}$. Assuming that the identity operator $I \in \mathcal{A}$, and that $\mathcal{A}$ contains the positive square root of each of its positive operators, one can prove as follows that $\mathfrak{M}$ is generated by its projections.

Recall first that any operator in $\mathcal{A}$ is a linear combination of unitary operators in $\mathcal{A}$ [1; 4, Proposition 3]. In fact, any operator in a $*$-algebra is a linear combination of self-adjoint contractions in that algebra, and if A is a self-adjoint contraction and $U = A + i(I - A^2)^{\frac{1}{2}}$, then U is unitary and $A = \frac{1}{2}(U + U^*)$.

Next we prove that any matrix $M_{ij}(A)$, with $A \in \mathcal{A}$ in the (i, j) location and 0 elsewhere, is for $i \neq j$ a linear combination of projections. If $U \in \mathcal{A}$ is unitary, denote by $N_{ij}(U)$ the matrix with $\frac{1}{2}U$ in the (i, j) location, $\frac{1}{2}U^*$ in the (j, i) location, $\frac{1}{2}I$ in the (i, i) and (j, j) locations, and 0 elsewhere, and by J_{ij} the matrix with I in the (i, i) and (j, j) locations and 0 elsewhere. Then $N_{ij}(U)$ and J_{ij} are evidently projections, and

$$M_{ij}(U) = \tfrac{1}{2}[2N_{ij}(U) - J_{ij}] + (i/2)[2N_{ij}(-iU) - J_{ij}].$$

Thus any matrix whose diagonal entries are linear combinations of projections is itself a linear combination of projections.

Finally, for any $A \in \mathcal{A}$ we have $M_{11}(A) = M_{12}(A)M_{21}(I)$. Similar computations for the other diagonal entries lead to the conclusion that $\mathfrak{M}$ is generated as an algebra by its projections. We remark that the last calculation could have been carried out using only Jordan operations, so that $\mathfrak{M}$ is also generated as a Jordan algebra by its projections.

The next step is to observe that certain von Neumann algebras may be regarded as $n \times n$ matrix algebras over some von Neumann algebra, and consequently are generated by their projections. In fact, let $\mathfrak{M}$ be a von Neumann algebra containing orthogonal equivalent projections $E_1, E_2, \cdots, E_n (n \geq 2)$ with $E_1 + E_2 + \cdots + E_n = I$. Then there exist partial isometries $U_j \in \mathfrak{M}$ with $U_j^* U_j = E_1$ and $U_j U_j^* = E_j$, $j = 1, 2, \cdots, n$. It is easy to verify that the mapping $M \to (U_i^* M U_j)$ is a $*$-isomorphism from $\mathfrak{M}$ onto the $n \times n$ matrix algebra over the von Neumann algebra $E_1 \mathfrak{M} E_1$. Therefore $\mathfrak{M}$ is generated by its projections.

Finally, this result and the structure theory for von Neumann algebras imply the following:

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