EQUIVALENT CONDITIONS FOR A RING TO BE A P-RING AND A NOTE ON FLAT OVERRINGS

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1. Introduction. In this paper a ring will mean a commutative ring with unity. An overring is a subring of the total quotient ring which contains the given ring. A ring R is a *P-ring* if every overring of R is integrally closed. P-ring is a generalization due to Davis [2] of Prüfer ring to rings which allow zero divisors. This paper provides equivalent conditions for a ring to be a P-ring. Also included are generalizations of some results of Richman [6] on flat overrings. An overring T of a ring R is a *flat overring* if T is a flat R-module. The notation and terminology used will be in general that of [8].

A Prüfer ring is an integral domain all of whose non-zero ideals are invertible. Krull [5] studied Prüfer ring and found several characterizations. In 1963 Jensen [4] established several ideal theoretic conditions equivalent to Prüfer ring. Davis [2] proved in 1964 that an integral domain R is a Prüfer ring if and only if every overring of R is integrally closed. In 1965 Richman [6] showed that a Prüfer ring R is characterized by the property that every overring of R is a flat overring.

The goal of this paper is to place a restriction on a commutative ring with unity which is weaker than requiring the ring to be an integral domain and to establish conditions, similar to those characterizing Prüfer ring, which are equivalent to the ring being a P-ring. Smith and Butts [7] have introduced a generalization of Prüfer ring different from P-ring and have studied conditions similar to those in this paper. However, under their generalization not all the conditions are equivalent.

2. Preliminary notions. The following definitions are due to Davis [2]. An ideal is a *0-ideal* if it consists entirely of zero divisors. A ring R is said to have *few zero divisors* if there is a finite number of maximal elements in the set of 0-ideals of R. Of the properties of rings R which have few zero divisors, we need the following (see [2] for proofs). (1) If $z \in R$ and x is a non-zero divisor in R, then there exists $u \in R$ such that $z + xu$ is a non-zero divisor. (2) If a is a non-zero divisor in an ideal A of R and $A = (z_i)$, then there exist (u_i) in R such that $A = (a, z_i + u_i a)$ and the $z_i + u_i a$ are non-zero divisors. Hence every regular ideal (ideal containing at least one non-zero divisor) can be generated by non-zero divisors.

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