NON-ANALYTIC FUNCTIONS OF TWO COMPLEX VARIABLES

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1. Introduction. We shall be concerned here with continuous complex-valued functions of two complex variables which are analytic in one variable. Such functions arise, for instance, when one considers the behavior of the roots of a complex polynomial as the coefficients vary continuously [5]. They also occur in the stability theory of differential equations [2].

The methods used are essentially topological in nature. The results needed from the theory of functions of one variable all have topological proofs which can be found in [6]. The theorems obtained are analogues of results in the classical theory. The main result is given in Theorem 4.

2. Definitions. We give the two following basic definitions:

Definition 1. A function f of two complex variables is said to belong to the family S at the point p if p is contained in an open bi-cylinder $B = (X, Y)$ such that $f \mid B$ is continuous and $f(x', y)$ is analytic on Y for each $x' \in X$.

Definition 2. A point p in the domain of f is called a singular point if the above conditions are not satisfied.

Since we consider only local properties, we assume that all functions involved are defined on all complex 2-space.

3. Invariance under differentiation. We now prove a theorem which will be needed later. f_y denotes the partial derivative.

Theorem 1. *If f belongs to S at p, then f_y belongs to S at p.*

Proof. Let $B = (X, Y)$ be the bi-cylinder containing p given in Definition 1. Let R be a disk such that $\bar{R} \subset Y$ and (X, R) contains p. Let x' be a point of X and let $\{x_n\}$ be any sequence of points converging to x'. Then by uniform continuity the family $\{f(x_n, y)\}$ converges uniformly to $f(x', y)$ on $\bar{R}$. Hence, the family $\{f_y(x_n, y)\}$ converges almost uniformly to $f_y(x', y)$ on R. It is known that uniform convergence is equivalent to continuous convergence in this case [3] and we see that if y' is any point in R and $\{y_n\}$ is any sequence of points converging to y', we have $f_y(x_n, y_n) \to f_y(x', y')$. Therefore, f meets the requirements of Definition 1 on (X, R).

4. Singularities. It is well known that if an analytic function of two (or more) complex variables has an isolated singularity, it is removable. Under our definitions, the same theorem holds.

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