REDUCING SUBSPACES OF ANALYTIC TOEPLITZ OPERATORS

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1. Introduction. It is elementary and well known that the simple unilateral shift has no nontrivial reducing subspaces [3; 40]. The smallest weakly closed algebra of operators containing the simple shift and the constants is the class of analytic Toeplitz operators. Which operators in this algebra have the same reducing subspaces as the shift? The object of this note is to give a sufficient condition (Theorem 1) for an analytic Toeplitz operator to have no nontrivial reducing subspaces. We shall also give two examples, one to illustrate the theorem and a second to show that our condition is not necessary. Finally we shall conclude with a sufficient condition (Theorem 2) for the existence of reducing subspaces.

Let μ be normalized Lebesgue measure on the Borel subsets of the unit circle $X = \{u: |u| = 1\}$ in the complex plane. The functions $e_n(u) = u^n$, $n = 0, \pm 1, \pm 2, \cdots$ constitute an orthonormal basis for $\mathcal{L}^2 = \mathcal{L}^2(\mu)$, and the functions $\{e_n : n \geq 0\}$ span the subspace $\mathcal{H}^2$ of $\mathcal{L}^2$. Let P be the orthogonal projection of $\mathcal{L}^2$ onto $\mathcal{H}^2$. A Laurent operator L_ϕ is a multiplication operator on $\mathcal{L}^2$ induced by a bounded measurable function ϕ on X:

$$L_\phi f = \phi f$$

for f in $\mathcal{L}^2$. A Toeplitz operator T_ϕ (or $T(\phi)$) is the compression of a Laurent operator L_ϕ to $\mathcal{H}^2$:

$$T_\phi = PL_\phi \mid \mathcal{H}^2 :$$

and T_ϕ is said to be analytic if ϕ is analytic, i.e. in $\mathcal{H}^2$.

2. Theorem 1. *Let T_ϕ be an analytic Toeplitz operator. If there exists a Borel subset M of X such that*

1) $\mu(M) > 0$,
2) *$\phi(M)$ and $\phi(X - M)$ are disjoint, and*
3) *the restriction of ϕ to M is one to one, then T_ϕ has no nontrivial reducing subspaces.*

Proof. By Lusin's Theorem [4; 242] there is a compact subset M_0 of M having positive measure on which ϕ is continuous. Since the restriction of ϕ to M_0 is one-to-one, and $\phi(M_0)$ and $\phi(X - M_0)$ are disjoint, there is no loss

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