THE EXCESS OF SETS OF COMPLEX EXPONENTIALS

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1. Purpose of this paper. We begin our discussion by reviewing some terminology [4], [8]. A set of complex exponentials $\{e^{i\lambda_n x}\}$ has *completeness interval* I if it is complete L^p on every interval of length less than I and on no larger interval. By convention $I = 0$ if the set is complete on no interval, and $I = \infty$ if it is complete on every finite interval. When a λ is repeated, we require a zero of corresponding multiplicity in the entire function $F(z)$ which vanishes at the λ_n's. This means that, when closure rather than completeness is in question, the functions

$$e^{i\lambda x}, xe^{i\lambda x}, \cdots, x^{m-1}e^{i\lambda x}$$

are available for the approximation. The set $\{e^{i\lambda_n x}\}$ is interpreted accordingly, though the question of multiple roots is not emphasized here (cf. Theorem 3).

The set has excess E on a closed interval if it remains complete when E terms $e^{i\lambda x}$ are removed, but not when $E + 1$ terms are removed. The deficiency is defined similarly, though we shall regard a deficiency as a negative excess. By convention $E = \infty$ if $I = \infty$, and $E = -\infty$ if $I = 0$. When $0 < I < \infty$, it is convenient to define $E = \infty$ if arbitrarily many terms $\{e^{i\lambda_n x}\}$ can be removed without losing completeness, and $E = -\infty$ if arbitrarily many terms can be adjoined without getting completeness. As is easily proved, when $E = -\infty$ one can actually adjoin infinitely many terms without getting completeness; hence, it is natural to surmise that when $E = \infty$, one can remove infinitely many terms without losing completeness. However, the truth or falsity of this conjecture is left as an open problem.

As is well known, the excess E is a far finer measure than is the completeness interval I. Thus, I is unchanged by adjunction or removal of a set of terms $\{e^{i\lambda_n x}\}$ for which

$$\sum \frac{1}{|\lambda_n|} < \infty$$

[8], while E is affected by removal or adjunction of a single term. Also, within wide limits, I is independent of the class of functions being considered. As an illustration, I for the class L is the same as I for the class of functions with continuous second derivatives. But E for the class L^p depends on p, and in general E increases by 1 if instead of $f \in L^p$ we require $f' \in L^p$. (These matters are discussed more fully in §2.)

If in addition to a sequence $\{e^{i\lambda_n x}\}$ we have a sequence $\{e^{i\mu_n x}\}$ we write $I(\lambda)$

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