ON THE LAW OF THE ITERATED LOGARITHM FOR CONTINUED FRACTIONS

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1. Introduction and results. Let θ be a real number between 0 and 1. Let $a_i(\theta) = a_i$, $i = 1, 2, 3, \cdots$, be the integral partial quotients of the simple continued fraction θ:

$$\theta = [a_0 ; a_1 , a_2 , \cdots]; \quad a_0 = 0. \tag{1}$$

If θ is a rational number, then the expansion (1) terminates, and θ can be represented uniquely as

$$\theta = [0; a_1 , a_2 ; \cdots , a_N]; \quad N \geq 1; \quad a_i \geq 1; \quad i = 1, 2, \cdots , N - 1; \quad a_N \geq 2.$$

In this case, we call θ rational of order N.

For $\theta = [0; a_1 , a_2 , \cdots]$ let

$$\theta_n(\theta) = \theta_n = [0; a_{n+1} , a_{n+2} , \cdots]. \tag{2}$$

If θ is rational of order N, define $\theta_i = 0$ for $i = N, N + 1, \cdots$. Notice that

$$a_{n+1} = \left[\frac{1}{\theta_n}\right], \tag{3}$$

where $[\alpha]$ denotes the greatest integer not exceeding α.

In 1935, A. Khintchine [9] proved the following remarkable

Theorem A. *For the continued fraction expansion $\theta = [0; a_1 , a_2 , \cdots]$ of almost all θ in* (0, 1) *we have*

$$\lim_{N\to\infty} \sqrt[N]{a_1 a_2 \cdots a_N} = K_1 , \tag{4}$$

where

$$\begin{aligned} K_1 &= \exp\left\{\frac{1}{\log 2} \sum_{k=2}^{\infty} (\log k) \log \left(1 + \frac{1}{k(k + 2)}\right)\right\} \\ &= \exp\left\{\frac{1}{\log 2} \int_0^1 \frac{\log [1/x]\, dx}{1 + x}\right\} = 2.68545 \cdots . \end{aligned} \tag{5}$$

The constant K_1, known as "Khintchine's Constant", has been calculated to at least 155 places [16]. For an ergodic theoretic proof of Theorem A, see

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