REMARK ON A PAPER OF S. K. BERBERIAN

By E. Durszt

In this Note we are going to give a negative answer to the following question of Berberian [1]: If T is an invertible operator in whose polar decomposition $T = UR$ the unitary factor $U = T(T^*T)^{-\frac{1}{2}}$ is "cramped", does it follow that 0 is not contained in the closure of the numerical range of T?

Let us recall that the *numerical range* of T is the set of complex numbers:

$$W(T) = \{(Tx, x) : ||x|| = 1\}.$$

We say that a unitary operator is *cramped* if its spectrum is contained in some open semicircle.

Theorem. *There exists an invertible operator T such that the unitary operator $U = T(T^*T)^{-\frac{1}{2}}$ is cramped and 0 is an inner point of $W(T)$.*

Proof. Let x_1 and x_2 be an orthonormal basis in the two-dimensional complex Euclidean space E^2. Choose a complex number ε such that

$$|\varepsilon| = 1 \quad \text{and} \quad 0 < \arg \varepsilon < \frac{\pi}{2},$$

and a real number r such that

$$\operatorname{Re} \varepsilon < r < 1.$$

Let U and R be the operators in E^2 with the matrices:

$$U = \begin{pmatrix} \varepsilon & 0 \\ 0 & \bar{\varepsilon} \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & r\bar{\varepsilon} \\ r\varepsilon & 1 \end{pmatrix},$$

in the basis $\{x_1, x_2\}$. It is clear that U is a cramped unitary operator. R is positive, because it is selfadjoint and its proper values are $1 + r$ (>0) and $1 - r$ (>0); both U and R have thus bounded inverses. So has

$$T = UR$$

a bounded inverse, too. T has the matrix

$$T = \begin{pmatrix} \varepsilon & r \\ r & \bar{\varepsilon} \end{pmatrix}$$

so we have:

$$(Tx_1, x_1) = ((\varepsilon x_1 + rx_2), x_1) = \varepsilon$$

$$(Tx_2, x_2) = ((rx_1 + \bar{\varepsilon} x_2), x_2) = \bar{\varepsilon},$$

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