ON FOX'S PROPERTY OF A SURFACE IN A 3-MANIFOLD

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1. Introduction. A theorem of Fox [2] with which we are concerned in this paper is as follows: Let F be a closed polyhedral surface without boundary with genus $g(F) > 0$ in a triangulated 3-sphere S. (Almost all arguments in the paper are from the semi-linear point of view and when point theoretic notions are involved, all spaces are separable metric.) Then there exists a simple closed polygonal curve c on F such that c does not bound a disk on F but bounds a polyhedral disk D in S in such a way that $c = \partial D = D \cap F$. A relation of the above theorem to the loop theorem [6] and Dehn's lemma [7] in a simply connected 3-manifold was suggested in [1]. The purpose of the paper is to generalize the theorem as follows:

Theorem A. *Let F be a closed orientable polyhedral surface without boundary with genus $g(F) > 0$ in the interior of a triangulated orientable 3-manifold M. If the injection homomorphism of the fundamental group $\Pi(F)$ into $\Pi(M)$, induced by the injection of F into M, is not an isomorphism, then there exists a polygonal simple closed curve c on F such that c does not bound a disk on F but bounds a polyhedral disk D in M in such a way that $D \cap F = \partial D = C$.*

Needless to say if such a surface F is in a 3-sphere S, then the homomorphism is trivial and is not an isomorphism and hence we have Fox's theorem. The proof will be given in §3 by using the loop theorem and Dehn's lemma.

We may say that a triangulated orientable 3-manifold M has *Fox's property* if for any closed orientable polyhedral surface F without boundary with genus $g(F) > 0$ there exists a polygonal simple closed curve c on F such that c does not bound a disk on F but bounds a disk D in M in such a way that $D \cap F = \partial D = C$. To characterize a 3-manifold in which Fox's property holds is the original problem asked of the author by Professor Harrold. It might be interesting to know that Fox's argument may be transferred directly to the case $S^2 \times S^1$ to show that his theorem is also true in the case. A partial answer for that question, a direct consequence of Theorem A, is as follows.

Theorem B. *Any orientable 3-manifold whose fundamental group is either finite or a finitely generated free group has Fox's property.*

We will use the following Theorem C about fundamental groups, together with the loop theorem and Dehn's lemma:

Theorem C. *Let X and Y be arcwise connected open subsets in $X \cup Y$ such that $X \cap Y$ is arcwise connected and the injections of $\Pi(X \cap Y)$ into $\Pi(X)$ and*

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