SUFFICIENT CONDITIONS FOR STABILITY OF A SOLUTION OF DIFFERENCE EQUATIONS

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1. Statement and discussion of results. If $x = (x_i)$ is an m-vector and $A = (a_{ij})$ is an $m \times m$ matrix, with complex elements, let $||x||^2 = \sum |x_i|^2$ and $||A|| = \max ||Ax||/||x||$ for $x \neq 0$. Also let $\rho(A) = |\lambda_1|$, $\sigma(A) = |\lambda_m|$, where $\lambda_1, \cdots, \lambda_m$ are the eigenvalues of A arranged so that $|\lambda_1| \geq \cdots \geq |\lambda_m|$. It is known that if A is constant and $\rho(A) < 1$, then $||x(t)|| \to 0$ as $t \to \infty$ for every solution $x(t)$ of the system of linear difference equations $x(t + 1) = Ax(t)$ in which t takes the values 0, 1, 2, 3, $\cdots$. Perron [8] and Hahn [2] have shown that the same is true of the solutions of the perturbed system

$$x(t + 1) = Ax(t) + f(t, x(t)), \tag{1}$$

provided that $||f(t, x)|| \leq \alpha \, ||x||$ with sufficiently small constant α. In proving results of this kind Hahn [2] and Kalman and Bertram [4] make use of a quadratic Lyapunov function x^*Px with

$$A^*PA - P = -Q, \tag{2}$$

where P, Q are positive definite hermitian matrices and $A^* = (\bar{a}_{ji})$ is the conjugate transpose of $A = (a_{ij})$. Hahn showed that if $\rho(A) < 1$, then (2) has a unique hermitian solution P for each hermitian Q and if Q is positive definite, then so is P. The eigenvalues $\rho(P)$, $\sigma(P)$, of P, are of some interest. It is shown in §2 that if Q is positive definite, then

$$\rho(P_0)\rho(Q) \geq \rho(P) \geq \rho(P_0)\sigma(Q), \qquad \sigma(P_0)\rho(Q) \geq \sigma(P) \geq \sigma(P_0)\sigma(Q), \tag{3}$$

where P_0 is the hermitian solution of (2) in the special case when Q is the unit $m \times m$ matrix I. That is,

$$A^*P_0A - P_0 = -I. \tag{4}$$

For $\rho(P_0)$, $\sigma(P_0)$ the following estimates are obtained.

THEOREM 1. *If* $\rho(A) < 1$, *then*

$$\rho(P_0) \leq (1 + ||A||)^{2m-2}(1 - |\lambda_1|^2)^{-1} \prod_{\nu=2}^{m} (1 - |\lambda_\nu|)^{-2}, \tag{5}$$

$$\rho(P_0) \geq (1 - |\lambda_1|^2)^{-1}, \tag{6}$$

$$\sigma(P_0) \leq (1 - |\lambda_m|^2)^{-1}, \tag{7}$$

$$\sigma(P_0) \geq (1 - \sigma(A^*A))^{-1}, \tag{8}$$

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