$\mathfrak{L}^1$-SPACES IN MEASURE ALGEBRAS OVER COMPACT SEMIGROUPS

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This paper is concerned with certain relationships between the structure of a compact topological semigroup S and the structure of the associated measure algebra $M(S)$ of all complex valued regular measures defined on the Borel sets, $B(S)$, of S. These measures form a Banach algebra under convolution product and total variation norm. Since S is compact, we may identify $M(S)$ with $C(S)^*$, the space of bounded linear functionals on $C(S)$, and will, at times, use the notation $\mu(f) = \int f(x)\, d\mu(x)$ for $f \varepsilon C(S)$, $\mu \varepsilon M(S)$. F. B. Wright [8] has investigated upper semicontinuous decompositions of a compact topological semigroup in terms of such a measure algebra.

A linear subspace of $M(S)$ will be called an ideal of $M(S)$ if it is closed under right and left convolution product with elements of $M(S)$. If $\mu \varepsilon M(S)$, $\mathfrak{L}^1(\mu)$ will denote those measures absolutely continuous with respect to μ. Clearly, $\mathfrak{L}^1(\mu)$ is a linear subspace of $M(S)$, and it is easy to see that if $\mu \geq 0$, then $\mathfrak{L}^1(\mu)$ is an ideal of $M(S)$ if and only if $x\mu$ and μx are elements of $\mathfrak{L}^1(\mu)$ for each point measure x determined by $x \varepsilon S$. The set of all $\lambda \varepsilon M(S)$ such that $\lambda \geq 0$, $||\lambda|| = 1$ is a weak-* compact topological semigroup under convolution multiplication and will be denoted $\tilde{S}$. The symbols K and $K(\tilde{S})$ will, respectively, denote the kernel of S and $\tilde{S}$; and for $\mu \varepsilon M(S)$, carrier (μ) will denote the compliment of the largest open set in S having $|\mu|$-measure zero.

Theorem 1. *If $\mu \varepsilon \tilde{S}$ and $\mathfrak{L}^1(\mu)$ is an ideal of $M(S)$ then carrier (μ) is an ideal of S.*

Proof. If $x \varepsilon S$, then the point measure x is an element of $\tilde{S}$ and since $\mathfrak{L}^1(\mu)$ is an ideal of $M(S)$, we have that $x\mu \ll \mu$ and $\mu x \ll \mu$. Now carrier $(x\mu) \subset$ carrier (μ) for if V is an open set having μ-measure zero, then $x\mu(V) = 0$. Hence, the definition of the carrier of a measure implies the desired inclusion. Similarly we have carrier $(\mu x) \subset$ carrier (μ). According to Wendel [7] [4], (x) carrier $(\mu) =$ carrier $(x\mu) \subset$ carrier (μ) and (carrier (μ)) $(x) =$ carrier $(\mu x) \subset$ carrier (μ). This completes the proof of the theorem.

Theorem 2. *If $\lambda^2 = \lambda \varepsilon \tilde{S}$ and $\mathfrak{L}^1(\lambda)$ is an ideal of $M(S)$, then:*

(A) *The kernel $K =$ carrier (λ).*
(B) $\lambda \varepsilon K(\tilde{S})$.

Proof. Since $\mathfrak{L}^1(\lambda)$ is an ideal of $M(S)$, Theorem 1 implies that carrier (λ) is an ideal of S. Since the kernel K is the minimal ideal, we have $K \subset$ carrier (λ).

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