ORDERS ON COMMUTATIVE RINGS

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The partially ordered rings that have been most thoroughly studied are totally ordered fields. The generalization to integral domains is natural, since every total order on an integral domain can be extended in a simple way to a total order on its field of fractions. We investigate the relationship between partial orders on a general commutative ring with identity and total orders on its residue class integral domains. This leads to the concept of symmetric cone. In the special case of a polynomial ring over a totally ordered field, we show that symmetric cones are induced by certain filters on the field. From this one can obtain a description of all possible total orders on a polynomial ring over a subfield of the real field. Finally, the fields that have a unique maximal order are determined.

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1. Preliminaries. All rings considered in the paper are assumed to be commutative rings with an identity. In the first two sections, the ring in any general discussion will be denoted by A. The field of rational numbers will be denoted by $\mathbf{Q}$, and the field of real numbers by $\mathbf{R}$. In treating partially ordered commutative rings, we find it convenient to work with the concept of positive cone, and to identify the order with the cone that defines it. This section is devoted to some general observations about cones. For further background on partially ordered rings, see [2].

We first recall the basic notions.

DEFINITION. A (positive) *cone* on the ring A is a subset P of A such that (1) $P + P \subseteq P$, (2) $PP \subseteq P$, and (3) $0 \notin P$. A *total cone* on A is a cone P on A such that (4) $P \cup \{0\} \cup -P = A$.

We note explicitly that a cone is allowed to be empty. On the other hand, if P is a nonempty cone on A, the set $P - P$ is a subring of A which contains P; and P induces a cone on $P - P$. Also, if P is any cone on A, conditions (1) and (3) imply that $P \cap -P = \varnothing$.

The family of all cones on a ring A is partially ordered by set inclusion, and it is inductive; thus, every cone can be embedded in a maximal cone. The following lemma is useful in studying maximality of cones (cf. [1; §8.4]).

LEMMA 1.1. *Let a be a nonzero element of the integral domain A, and let P*

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