EQUATIONAL CLASSES OF LATTICES

By G. Grätzer

In his paper [2] B. Jònsson considers the lattice of all equational classes of lattices. The zero element of this lattice is the class **0** of all one-element lattices. This is covered by the class **D** of all distributive lattices, and no other class covers **0**. Let M_5 and N_5 denote the five element modular and non-distributive, and the five element non-modular lattice, respectively, $\mathbf{M}_5$ and $\mathbf{N}_5$ the equational classes generated by them. Then it is easy to see that **D** is covered by exactly two classes, $\mathbf{M}_5$ and $\mathbf{N}_5$.

One of the problems raised by B. Jònsson in [2] is to find how many equational classes of modular lattices cover $\mathbf{M}_5$.

Let **K** be an equational class of lattices, $S(\mathbf{K})$ the class of all subdirectly irreducible lattices in **K**. It is easy to see that $S(\mathbf{K})$ generates **K** and thus $\mathbf{K}_1 = \mathbf{K}_2$ if and only if $\mathbf{S}(\mathbf{K}_1) = \mathbf{S}(\mathbf{K}_2)$.

Now suppose that **K** covers $\mathbf{M}_5$ and contains modular lattices only. Then $S_i(\mathbf{K})$ contains at least three lattices L, M_5 and 2 (the two element lattice) up to isomorphism. Indeed, since $\mathbf{K} \supset \mathbf{M}_5$, $S_i(\mathbf{K}) \supset S_i(\mathbf{M}_5) \supseteq \{M_5 , 2\}$. Therefore there exists a lattice L with $L \in S_i(\mathbf{K})$, $L \notin S_i(\mathbf{M}_5)$. Since L is non-distributive it contains sublattices isomorphic to M_5 and 2. Therefore the class generated by L properly contains M_5 , thus L generates **K**. Thus we proved the following statement:

Let **K** be an equational class of modular lattices, covering $\mathbf{M}_5$. Then **K** can be generated by a single modular lattice L, which is subdirectly irreducible.

My aim in this note is to find all finite subdirectly irreducible lattices which generate an equational class covering $\mathbf{M}_5$.

Theorem. *Let L be a finite modular subdirectly irreducible lattice. L generates an equational class covering* $\mathbf{M}_5$ *if and only if L is isomorphic to one of the two lattices of* Fig. 1.

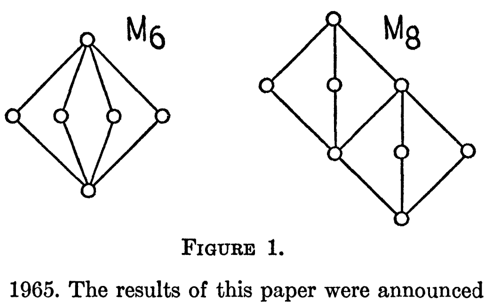

Figure 1.

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