THE THEORY OF MULTIVALENT FUNCTIONS

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1. Introduction. Various criteria for multivalence of analytic functions have been obtained by several authors [1], [5], [6], [7], [13]. In a recent note K. Sakaguchi obtained the following result [13].

Theorem A. *Let $f(z) = z^p + \cdots$, $\varphi(z)$ be regular in the disk $\bar{E}(|z| \leq r)$ and the plane $|z| < +\infty$ respectively, and $f'(z) \neq 0$ for the annulus $0 < |z| \leq r$ (hereafter called A). If neither $f(z)$ nor $\varphi'(\log f(z))$ vanishes on the circle $|z| = r$ (hereafter called C_z) and the inequality*

$$\int_C d \arg d\varphi (\log f(z)) > -\pi$$

for all arcs C on C_z, then $f(z)$ is p-valent in $\bar{E}$.

In §2 of this paper we shall show a generalization of the above theorem. In §3, some sufficient conditions for starlikeness or convexity of analytic functions will be studied. In §4, meromorphic close-to-convex functions will be considered, and we shall study the application of results in §5.

2. Fundamental results.

Definition 1. Let $f(z)$ be regular in $\bar{E}$ and $f'(z) \neq 0$ on C_z and let C_w be the image curve of C_z by the function $w = f(z)$. When a directed arc C_w' on C_w is a simple closed curve whose interior does not contain the origin and whose direction is clockwise, we say that C_w' makes a loop [15; 213] on C_w. On such a loop C_w' we have

$$\int_{C_w'} d \arg df(z) \leq -\pi.$$

Lemma 1. *Let $f(z)$ be regular for $\bar{E}$ and $f'(z) \neq 0$ on C_z. Suppose that $\int_0^{2\pi} d \arg df(z) = 2k\pi$ (k: positive integer). If $f(z)$ is $(p + 1)$-valent in $E(|z| < r)$, then we have at least $p + 1 - k$ arcs $C_1, C_2, \cdots, C_{p+1-k}$ on C_z which have no common parts except perhaps the ends of them for which*

$$\int_{C_i} d \arg df(z) \leq -\pi, \qquad i = 1, 2, \cdots, p + 1 - k$$

hold.

We owe this lemma to T. Umezawa [15; 224–225].

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