# INTEGRABILITY OF ULTRASPHERICAL SERIES

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The ultraspherical polynomials $C_n^\lambda$, for fixed $\lambda > 0$, may be defined on $-1 \le x \le 1$ by the expansion $(1 - 2xz + z^2)^{-\lambda} = \sum_{n=0}^\infty C_n^\lambda(x) z^n$ for $|z| < 1$. The polynomials $t_n^\lambda C_n^\lambda$, with

$$t_n^\lambda = \left\{ \frac{\Gamma(\lambda)\Gamma(2\lambda)}{\sqrt{\pi}\,\Gamma(\lambda + \frac{1}{2})} \frac{(n + \lambda) n!}{\Gamma(n + 2\lambda)} \right\}^{\frac{1}{2}} = O(n^{1-\lambda}),$$

are orthonormal on $[-1, 1]$. [4; (144.27) and (144.28)]. If $f \varepsilon L([-1, 1]; d\sigma)$, where $d\sigma(x) = (1 - x^2)^{\lambda - \frac{1}{2}}\, dx$, the ultraspherical coefficients of $f$ may be defined: $a_n = t_n^\lambda \int_{-1}^1 f(x) C_n^\lambda(x)\, d\sigma(x)$. If the series $\sum_{n=0}^\infty a_n t_n^\lambda C_n^\lambda(x)$ is summable on $[-1, 1]$, then it is summable to the value $f(x)$. [5; Theorem 9.1.4 with $\alpha = \beta = \lambda - \frac{1}{2}$]. An asymptotic formula with $x = \operatorname{Cos} \theta$, [5; (8.21.14)]

$$(1) \quad C_n^\lambda(\operatorname{Cos} \theta) = \frac{2\Gamma(n + \lambda)}{\Gamma(\lambda) n!} \sum_{k=0}^{m-1} \frac{\Gamma(k + \lambda)}{\Gamma(\lambda) k!} \frac{(1 - \lambda)(2 - \lambda) \cdots (k - \lambda)}{(n - 1 + \lambda)(n - 2 + \lambda) \cdots (n - k + \lambda)}$$
$$\cdot (2 \operatorname{Sin} \theta)^{-k-\lambda} \operatorname{Cos}\left[ (n - k + \lambda)\theta - (k + \lambda)\frac{\pi}{2} \right] + O(n^{\lambda - m - 1}), \quad \epsilon \le \theta \le \pi - \epsilon,$$

shows that the function of $\theta$, $t_n^\lambda (\operatorname{Sin} \theta)^\lambda C_n^\lambda(\operatorname{Cos} \theta)$ behaves roughly like a linear combination of $\operatorname{Cos} n\theta$ and $\operatorname{Sin} n\theta$.

Theorems 1 and 2 below are analogous to the following results of R. P. Boas, Jr. [1] concerning the integrability of trigonometric series.

A. For $g \varepsilon L(0, \pi)$ let either $b_n = \int_0^\pi g(\theta) \operatorname{Sin} n\theta\, d\theta$ or $b_n = \int_0^\pi g(\theta) \operatorname{Cos} n\theta\, d\theta$. In either case suppose $b_n \ge 0$ and $0 < \beta < 1$. If $\int_0^\pi \theta^{\beta - 1} |g(\theta)|\, d\theta < \infty$, then $\sum_1^\infty n^{-\beta} b_n < \infty$.

B. Suppose $b_n - b_{n+2} \ge 0$, $b_n \to 0$, and $0 < \beta < 1$. Put either

$$g(\theta) = \sum_1^\infty b_n \operatorname{Sin} n\theta \quad \text{or} \quad g(\theta) = \sum_1^\infty b_n \operatorname{Cos} n\theta.$$

In either case if

$$\sum_1^\infty n^{-\beta} b_n < \infty, \quad \text{then} \quad \int_0^\pi (\operatorname{Sin} \theta)^{\beta - 1} |g(\theta)|\, d\theta < \infty.$$

THEOREM 1. *Let $f \varepsilon L([-1, 1]; d\sigma)$ and put*

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