A DIRECT INTEGRAL CONSTRUCTION

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Introduction. The object of this paper is to give a fairly concise construction of the von Neumann [3] direct integral decomposition associated with the functional calculus generated by a family of commuting self-adjoint operators acting on a separable hilbert space. The construction is reduced to an application of the Dunford–Pettis theorem and with this, most of the measure-theoretic complexities usually associated with this result vanish. The version of the Dunford–Pettis theorem that we need is the following:

Theorem. *Assume given a separable banach space D, a measure v whose support is contained in a locally compact space V and a bilinear map G of $D \times D$ into $L^\infty(v)$. Then, for all λ in V, there exists a bilinear map G_λ of $D \times D$ into the complex numbers such that for any pair $a, b \in D$, $G(a, b)(\lambda) = G_\lambda(a, b)$ almost everywhere* (a.e.) (cf., e.g., [2], *exposé no.* 4).

We will assume that the spectral theorem is given in terms of the spectral family [1; 328], namely, that for a certain separable hilbert space H, we are given a bilinear map G carrying $H \times H$ into $L^1(m)$, where m is a positive and bounded measure with support in V. For arbitrary x, y in H, G must have the following properties

1. $G(x, y) = \overline{G(y, x)}$
2. $G(x, x) \geq 0$
3. $\int G(x, y)\, dm = (x, y)_H$
4. $|G(x, y)|_{L^1} \leq |x|_H\, |y|_H$
5. For f in $L^\infty(m)$, a bounded operator T_f on H is defined by $(T_f x, y) = \int fG(x, y)\, dm$. We also require $G(T_f x, y) = f \cdot G(x, y)$.

The construction. Let D be any hilbert space dense in H with an orthonormal basis $\{a_n\}_{n=1,\dots}$ for which $\sum_1^\infty |a_n|_H < \infty$ and the norm of the map of D into H is ≤ 1. That such exists is clear. If $\{x_n\}_{n=1,\dots}$ is an orthonormal basis for H, we put $a_n = x_n/n^2$ and define $(a_n, a_m)_D = \delta_{nm}$. This extends in an obvious way to a unique inner product over the linear span of the a_n, which we will denote by D_0, and the completion of D_0 gives us the desired D.

Lemma 1. *There exists an f in $L^1(m)$ for which $|G(a, b)| \leq f$ for all a, b in the unit ball of D.*

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