INEQUALITIES IN QUADRATIC FORMS

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Introduction. Let $A_1, \ldots, A_m$ be mutually commuting symmetric matrices, and let G be a semiconvex function defined in E_m. In this paper we deal with some inequalities which connect the quadratic form of $G(A_1, \cdots, A_m)$ with the quadratic forms of $A_1, \cdots, A_m$.

§1 contains definitions, notations and the statement of some general facts.

In §2 we consider the expression

$$\psi(x) = \frac{G(A_1, \cdots, A_m)(x, x)}{(x, x)} - G\left(\frac{A_1(x, x)}{(x, x)}, \cdots, \frac{A_m(x, x)}{(x, x)}\right),$$

where $A(x, x)$ denotes the quadratic form corresponding to A. Let D be an open set on the unit sphere which fulfills some conditions determined by $A_1, \cdots, A_m$. We prove that if $\psi(x) \geq 0$ on the boundary of D, then $\psi(x) \geq 0$ also in D (Theorem 1). For the particular case $G(v_1, v_2) = v_1 v_2$, Theorem 1 reduces to a statement which generalizes an inequality of Tchebychef [4; 43].

In §3 we derive from Theorem 1 an inequality of Mulholland and Smith [8]. Applying this inequality, we show that certain sequences, which converge to the greatest characteristic value of a nonnegative symmetric matrix, are monotonic increasing. Next, we examine several possibilities for generalizing the inequality of Mulholland and Smith (Theorem 2).

1. Definitions and notations. All the vectors concerned in this paper are real, and the matrices are real and square. Matrices are denoted by capital Latin letters and vectors by small ones. A matrix $A = (a_{ij})$ is positive ($A > 0$) or nonnegative ($A \geq 0$) if, for every i and j, $a_{ij} > 0$ or $a_{ij} \geq 0$ respectively. We use similar definitions and notations for positive and nonnegative vectors. The inner product of two vectors x and y of the real n-dimensional Euclidean space E_n is denoted by (x, y). S_{n-1} is the unit sphere of E_n; i.e. $S_{n-1} = \{x \mid x \in E_n, (x, x) = 1\}$.

Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix. The quadratic form $A(x, x)$ corresponding to A is defined by

$$A(x, x) = (Ax, x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j .$$

A is positive definite (nonnegative definite) if $A(x, x) > 0$ ($A(x, x) \geq 0$) for every $x \neq 0$.

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