A NOTE ON MATRICES OVER EXTENSION FIELDS

BY DAVID CARLSON

In [1] this author noted that a square complex matrix A is similar to a real matrix if and only if $A = H_1 H_2$ for some Hermitian matrices H_1 and H_2 (part of this result may be found in [9; 63], or [8]). In this note we discuss several aspects of a more general question: given fields E and F, $E \subseteq F$, when is a square matrix over F similar to a matrix over E?

1. Let E, F be given, together with an automorphism φ of F with E as fixed field. We denote by $M_m(F)$ the set of all $m \times m$ matrices over F. For $A = [a_{ij}] \in M_m(F)$ we define $A^\varphi = [\varphi(a_{ij})]$. We have

$$(AB)^\varphi = A^\varphi B^\varphi, \qquad (A^\varphi)^{-1} = (A^{-1})^\varphi, \qquad (A^\varphi)^T = (A^T)^\varphi.$$

Let $f(x) = \sum_{i=0}^{r} a_i x^i \in F[x]$; we define $f^\varphi(x) = \sum \varphi(a_i) x^i$.

PROPOSITION. *$A \in M_m(F)$ is similar to a matrix over E if and only if A is similar to A^φ.*

Proof. If $A = RBR^{-1}$ for some $B \in M_m(E)$, then

$$A^\varphi = R^\varphi B (R^\varphi)^{-1} = (R^\varphi R^{-1}) A (R^\varphi R^{-1})^{-1}.$$

To prove the converse, suppose A has similarity invariants $h_1(x), \cdots, h_k(x)$. Then A^φ has similarity invariants $h_1^\varphi(x), \cdots, h_k^\varphi(x)$. If A is similar to A^φ, we must have $h_i^\varphi(x) = h_i(x) \in E[x]$, $i = 1, \cdots, k$, so that A is similar to $\sum \oplus C(h_i(x)) \in M_m(E)$, where $C(h_i(x))$ is the companion matrix of $h_i(x)$. The proof is complete.

2. Next we specialize to the case where $[F : E] = 2$. We write $(A^\varphi)^T = A^*$ and say that A is φ-Hermitian if $A^* = A$. We can now generalize the remark at the beginning of this note.

COROLLARY. *Let $A \in M_m(F)$. If $[F : E] = 2$, the following are equivalent:*

(i) *A is similar to a matrix over E,*

(ii) *there exists a φ-Hermitian H for which $H^{-1}AH = A^*$,*

(iii) *there exist φ-Hermitian H_1 and H_2, H_2 nonsingular, for which $A = H_1 H_2$.*

Proof. (i) $\Rightarrow$ (ii). Suppose $R^{-1}AR = B \in M_m(E)$. By a theorem of Taussky and Zassenhaus [10], there exists a symmetric $S \in M_m(E)$ for which $S^{-1}BS = B^T = B^*$. Calculation shows that

$$(RSR^*)^{-1} A (RSR^*) = A^*;$$

as $[F : E] = 2$, φ^2 is the identity automorphism and RSR^* is φ-Hermitian.

Received August 2, 1965. Research supported by a Directed Research Grant from Oregon State University and by NSF Grant GP-4051.