THE LEXICOGRAPHIC PRODUCT OF GRAPHS

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1. In [1] Sabidussi gave necessary and sufficient conditions for $G(X \circ Y) = G(X) \circ G(Y)$ ("$\circ$" denotes the lexicographic product on graphs and the wreath product on groups) if X and Y are from appropriate classes of graphs. In [2] he extended this result by enlarging the classes of graphs under consideration. We now complete the problem by proving the following:

THEOREM. *For graphs X and Y, $G(X \circ Y) = G(X) \circ G(Y)$ if and only if*

(1) *Y is connected if $R \neq \Delta$*
(2) *Y' is connected if $S \neq \Delta$.*
(3) *If Y has a set of vertex disjoint section graphs $\{Y_\alpha\}_{\alpha \in \Omega}$, $|\Omega| \geq 2$, such that $Y_\alpha \approx Y$ for all $\alpha \in \Omega - \{1\}$, $1 \in \Omega$, $V(Y) = \bigcup_{\alpha \in \Omega} V(Y_\alpha)$, and for $\alpha, \beta \in \Omega$ either all or none of the possible edges between Y_α and Y_β exist in Y, then X does not contain a section graph T on $V(T) = \{x_\alpha\}_{\alpha \in \Omega}$ such that* (a) *$V(X, x_\alpha) - V(T) = V(X, x_\beta) - V(T)$ for all $\alpha, \beta \in \Omega$,* (b) *$[x_\alpha, x_\beta] \in E(X)$ if and only if $[y_\alpha, y_\beta] \in E(Y)$ for some $y_\alpha \in V(Y_\alpha)$ and $y_\beta \in V(Y_\beta)$, and* (c) *$[x_1, x_\alpha] \in E(X)$ for all $\alpha \in \Omega - \{1\}$.*

By a graph X we mean a set $V(X)$ (called the vertices of X) together with a set $E(X)$ (called the edges of X) of unordered pairs of *distinct* elements of $V(X)$. Unordered pairs will be denoted by brackets. For $x \in V(X)$ we will usually write $x \in X$. For a set A, $|A|$ denotes the cardinality of A and by $|X|$ we mean $|V(X)|$. For $x \in X$ we put $V(X, x) = \{x' \in V(X) : [x, x'] \in E(X)\}$ and $d(X, x) = |V(X, x)|$. R and S are equivalence relations on $V(X)$ if we set xRx' if and only if $V(X, x) = V(X, x')$ and xSx' if and only if $V(X, x) \cup \{x\} = V(X, x') \cup \{x'\}$. $\Delta = \{(x, x) : x \in V(X)\}$. By the complement of a graph X we mean the graph X' given by $V(X') = V(X)$ and $E(X') = \{[x, x'] : x, x' \in V(X), x \neq x', [x, x'] \notin E(X)\}$. By $G(X)$ we denote the set of all permutations φ of $V(X)$ such that $[\varphi x, \varphi x'] \in E(X)$ if and only if $[x, x'] \in E(X)$. We say that a graph T is a section graph of X on $V(T)$ if $V(T) \subseteq V(X)$ and $E(T) = \{[x, x'] \in E(X) : x, x' \in V(T)\}$. The lexicographic product of two graphs X and Y is the graph $X \circ Y$ given by $V(X \circ Y) = V(X) \times V(Y)$ and $E(X \circ Y) = \{[(x, y), (x', y')] : [x, x'] \in E(X)$ or $x = x'$ and $[y, y'] \in E(Y)\}$. If G and H are permutation groups on sets A and B respectively, then the wreath product of G and H is the group $G \circ H$ of all permutations f on $A \times B$ for which there exist $g \in G$ and $h_a \in H$ for each $a \in A$ such that $f(a, b) = (ga, h_a b)$ for all $(a, b) \in A \times B$.

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