CURVILINEAR OSCILLATIONS OF HOLOMORPHIC FUNCTIONS

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Let f be a function with domain the open unit disc $D : |z| < 1$ and range in the Riemann sphere Ω. We say that the curve $\tau \subset D$ is an arc at 1 if $\tau \cup \{1\}$ is a Jordan arc. For an arc τ at 1, let C_τ denote the set $\{a \in \Omega$: there exists a sequence $\{z_n\} \subset \tau$ such that $z_n \to 1$ and $f(z_n) \to a\}$ of cluster values of f, on τ at 1, and let $|C_\tau|$ denote the diameter (in the chordal metric $\chi(a, b)$) of C_τ. It is clear that f has a limit at 1 on the arc τ at 1 if and only if $|C_\tau| = 0$. We prove

THEOREM 1. *If f is holomorphic in D, then either there exists an arc τ at 1 such that $|C_\tau| = 0$ or there exists a positive number h such that if τ is any arc at 1, then $|C_\tau| \geq h$.*

Remark. We emphasize that h depends only on f and the point 1 and is independent of the arc τ at 1.

Proof. Suppose that there does not exist a positive number h such that if τ is an arc at 1, then $|C_\tau| \geq h$. Let $\{\tau_n\}$ be a sequence of arcs at 1 such that $|C_n| \to 0$, where $C_n = C_{\tau_n}$. By choosing a subsequence of $\{\tau_n\}$ if necessary, we may suppose that there exists $a \in \Omega$ (possibly $a = \infty$) such that

$$|\{a\} \cup C_n| \to 0 (n \to \infty). \tag{1}$$

Since there are only countably many points of Ω that are projections of branch points of the Riemann surface of f over Ω, we can choose $\{r_n\}$ such that $0 < r_{n+1} < r_n (n \geq 1)$, $r_n \to 0$, and each component of $\{z : \chi(f(z), a) = r_n\}$ has no multiple points. Let

$$\Delta_n = \left\{|z - 1| < \frac{1}{n}\right\} \cap \{z : \chi(f(z), a) < r_n\}.$$

By noting (1) and again choosing a subsequence of $\{\tau_n\}$ if necessary, we may suppose that

$$C_n \subset \{w \in \Omega : \chi(w, a) < r_n\};$$

and by choosing subarcs (that are arcs at 1) of the arcs τ_n, we may suppose that $\tau_n \subset \Delta_n$. Then since $\Delta_{n+1} \subset \Delta_n (n \geq 1)$, we have

$$\tau_m \subset \Delta_n \qquad (m \geq n). \tag{2}$$

It must be the case that either

(3) for each n there are at most two components Δ of Δ_n such that there exists $m > n$ such that $\tau_m \subset \Delta$,

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